# Reactions Theory I

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Overview of Nuclear Reactions

Compound and Direct Reactions

Types of direct reactions

Overview of Nuclear Reactions

- Compound and Direct Reactions
- Types of direct reactions

Elastic Cross Sections

- Phase Shifts from Potentials
- Integral Expressions

# Classification by Outcome

1. **Elastic scattering:**
   projectile and target stay in their g.s.
2. **Inelastic scattering:**
   projectile or target left in excited state
3. **Transfer reaction:**
   1 or more nucleons moved to the other nucleus
4. **Fragmentation/Breakup/Knockout:**
   3 or more nuclei/nucleons in the final state
5. **Charge Exchange:**
   A is constant but Z (charge) varies, e.g. by pion exchange
6. **Multistep Processes:**
   *intermediate* steps can be any of the above
   ('virtual' rather than 'real')

7. **Deep inelastic collisions:**
   Highly excited states produced
8. **Fusion:**
   Nuclei stick together
9. **Fusion-evaporation:**
   fusion followed by particle-evaporation and/or gamma emission
10. **Fusion-fission:**
    fusion followed by fission

The first 6 processes are *Direct Reactions* (DI)
The last 3 processes give a *Compound Nucleus* (CN).

# Compound and Direct Reactions

So when two nuclei collide there are 2 types of reactions:

1. Nuclei can coalesce to form highly excited **Compound nucleus (CN)** that lives for relatively long time.
   Long lifetime sufficient for excitation energy to be shared by all nucleons. If sufficient energy localised on one or more nucleons (usually neutrons) they can escape and CN decays.
   Independence hypothesis: CN lives long enough that it loses its memory of how it was formed. So probability of various decay modes independent of entrance channel.
2. Nuclei make 'glancing' contact and separate immediately, said to undergo **Direct reactions(DI)**.
   Projectile may lose some energy, or have one or more nucleons transferred to or from it.

# Location of reactions:

CN reactions at small impact parameter,

DI reactions at surface & large impact parameter.

CN reaction involves the whole nucleus.

DI reaction usually occurs on the surface of the nucleus. This leads to diffraction effects.

**Duration of reactions:**

A typical nucleon orbits within a nucleus with a period of $\sim 10^{-22}$ sec. [as K.E. $\sim$ 20 MeV].

If reaction complete within this time scale or less then no time for distribution of projectile energy around target $\Rightarrow$ DI reaction occurred. CN reactions require $\gg 10^{-22}$ sec.

# Angular distributions:

In DI reactions differential cross section strongly forward peaked as projectile continues to move in general forward direction.

Differential cross sections for CN reactions do not vary much with angle (not complete isotropy as still slight dependence on direction of incident beam).

# Types of direct reactions:

Can identify various types of DI processes that can occur in reactions of interest:

1. **Elastic scattering**: $A(a,a)A$ – zero Q-value — internal states unchanged.

2. **Inelastic scattering**: $A(a,a')A^*$ or $A(a,a^*)A^*$. Projectile $a$ gives up some of its energy to excite target nucleus $A$. If nucleus $a$ also complex nucleus, it can also be excited.

   [If energy resolution in detection of $a$ not small enough to resolve g.s. of target from low-lying excited states then cross section will be sum of elastic and inelastic components. This is called **quasi-elastic scattering**].

3. **Breakup reactions**: Usually referring to breakup of projectile *a* into two or more fragments. This may be **elastic** breakup or **inelastic** breakup depending on whether target remains in ground state.

4. **Transfer reactions**:
   Stripping:
   Pickup:

5. **Charge exchange reactions**: mass numbers remain the same. Can be elastic or inelastic.

# Some terminology

**Reaction channels:**

In nuclear reaction, each possible combination of nuclei is called a **partition**.

Each partition further distinguished by state of excitation of each nucleus and each such pair of states is known as a **reaction channel**.

The initial partition, $a + A$ (both in their ground states) is known as the incident, or entrance channel. The various possible outcomes are the possible exit channels.

In a particular reaction, if not enough energy for a particular exit channel then it is said to be closed.

# Spherical Potentials

Non-relativistic, 2-body formalism of Schrödinger equation (SE).
Look at 2-body system in potential $V(r)$

$$\begin{aligned} \mathbf{r} &= (\mathbf{r}_1 - \mathbf{r}_2) \\ \mathbf{R} &= (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2) \end{aligned}$$

The time-indendent Schrödinger equation is

$$\hat{H}\Psi = E\Psi \tag{1}$$

The Hamiltonian for the system is

$$\begin{aligned} \hat{H} &= -\frac{\hbar^2}{2m_1}\nabla_{\mathbf{r1}} - \frac{\hbar^2}{2m_2}\nabla_{\mathbf{r2}} + V(r) \\ &= -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}} - \frac{\hbar^2}{2m}\nabla_{\mathbf{r}} + V(r) \end{aligned} \tag{2}$$

[ $m = m_1 m_2/(m_1 + m_2)$ and $M = m_1 + m_2$ ]

Thus can look for separable solutions of the form

$$\Psi(\mathbf{R},\mathbf{r}) = \phi(\mathbf{R})\psi(\mathbf{r}) \tag{3}$$

Substituting for $\Psi$ back in SE (1) gives LHS function of **R** and RHS function of **r**. Thus both equal to common constant, $E_{cm}$. Hence

$$-\ \frac{\hbar^2}{2M}\nabla^2_{\mathbf{R}}\ \phi(\mathbf{R})\ =\ E_{cm}\ \phi(\mathbf{R}) \tag{4}$$

and

$$\left(-\ \frac{\hbar^2}{2m}\nabla^2_{\mathbf{r}} + V(r)\right)\psi(\mathbf{r})\ = E_{rel}\ \psi(\mathbf{r}) \tag{5}$$

where $\quad E_{rel}\ =\ E\ -\ E_{cm}.$

In scattering, if $m_1$ is projectile incident on stationary target $m_2$ then

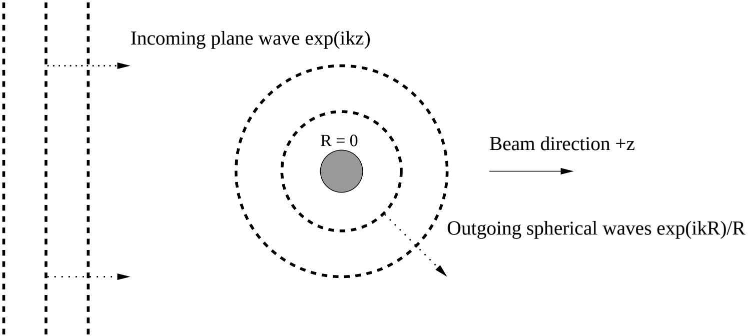
$$E_{cm} = \frac{m_1}{m_1 + m_2} \; E$$

$$E_{rel} = \frac{m_2}{m_1 + m_2} \; E$$

Solution to (4) is simple: $\phi(\mathbf{R}) = A\, e^{i\mathbf{K}\cdot\mathbf{R}}$ which is plane wave. Thus c.o.m. moves with constant momentum $\hbar\mathbf{K}$ and does not change after scattering. (Note, $E_{cm} = \hbar^2 K/2M$).

The real physics is in Eq.(5).

# Spherical Potentials: $\psi(\mathbf{r})$ from $V(r)$




If incident beam $\sim$ 1 cm wide, this is $10^{13}$ fm $= 10^{12} \times$ nuclear size).

Thus beam can be represented by **plane wave** $e^{i\mathbf{k}\cdot\mathbf{r}}$

As $|\mathbf{r}| \to \infty$ (i.e. moving away radially from scattering centre),

$$\psi(\mathbf{r}) \;\to\; N \left( e^{i\mathbf{k}\cdot\mathbf{r}} \;+\; f(\theta,\varphi)\, \frac{e^{ikr}}{r} \right) \tag{6}$$

where $k$ is defined as $E_{rel} = \hbar^2 K/2m$. Take $N = 1$.

In QM, flux (probability current density) is given by

$$\mathbf{J} = Re\left[\psi^* \left( -\frac{i\hbar}{m} \nabla_{\mathbf{r}} \right) \psi \right]$$

For incident flux, $\psi_{inc} = e^{i\mathbf{k}\cdot\mathbf{r}}$ and

$$\begin{aligned} \mathbf{J}_{inc} &= Re\left[ e^{-i\mathbf{k}\cdot\mathbf{r}} \left( -\frac{i\hbar}{m} \nabla_{\mathbf{r}} \right) e^{i\mathbf{k}\cdot\mathbf{r}} \right] \\ &= \frac{\hbar\mathbf{k}}{m} \;. \end{aligned} \tag{7}$$

# Cross Section

For scattered flux $\psi_{scat} = f(\theta, \varphi)\ \frac{e^{ikr}}{r}$ and hence we obtain

$$\mathbf{J}_{scat} = \mathrm{J}_{inc}\ \frac{|f(\theta,\varphi)|^2}{r^2}\ \hat{\mathbf{r}} \tag{8}$$

Define the **differential cross-section** (in units of area) as

The number of particles scattered into unit solid angle per unit time, per unit incident flux, per target point,

so

$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2. \tag{9}$$

# What does $f(\theta, \varphi)$ look like?

We know what the solution must look like asymptotically:

$$\psi(\mathbf{r}) \;\rightarrow\; N \left( e^{i\mathbf{k}\cdot\mathbf{r}} \;+\; f(\theta,\varphi)\, \frac{e^{ikr}}{r} \right) \tag{10}$$

For $V(r)$ a central potential, choose partial wave solutions

$$\psi(\mathbf{r}) = \sum_{\ell} \frac{u_\ell(r)}{kr}\, Y_{\ell 0}(\theta) \tag{11}$$

and choose $z$-axis parallel to incident beam, so $e^{i\mathbf{k}\cdot\mathbf{r}} \;=\; e^{ikz}$.

Then can write radial Schrödinger equation as

$$\frac{d^2 u_\ell}{dr^2} + \left[ k^2 - \frac{2m}{\hbar^2} V(r) - \frac{\ell(\ell+1)}{r^2} \right] u_\ell = 0 \tag{12}$$

# Asymptotic Solutions

Choose $V(r) = 0$ for $r > r_0$. Beyond $r_0$ get **free solution**

$$u''_\ell + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell = 0 \qquad (13)$$

Free solution is related to Coulomb functions

$$\begin{array}{llllll} r > r_0 : & u_\ell & = & A_\ell & F_\ell(kr) & + B_\ell \ G_\ell(kr) \\ & & & & \uparrow & \uparrow \\ & & & & kr\, j_\ell(kr) & -kr\, n_\ell(kr) \\ & & & & \text{(regular)} & \text{(irregular)} \\ r \to \infty : & & & & \to \sin(kr - \ell\pi/2) & \cos(kr - \ell\pi/2) \end{array} \qquad (14)$$

# Phase Shifts

As $r \to \infty$

$$\begin{aligned} u_\ell &\to A_\ell \sin(kr - \ell\pi/2) + B_\ell \cos(kr - \ell\pi/2) \\ &= C_\ell \sin(kr - \ell\pi/2 + \delta_\ell) \end{aligned} \quad (15)$$

where $\delta_\ell$ is known as the **phase shift**, so $\tan \delta_\ell = B_\ell / A_\ell$ here.

If $V = 0$ then solution must be valid everywhere, even at origin where it has to be regular. Thus $B_\ell = 0$.

So, asymptotically (long way from scattering centre):

$$\begin{aligned} \text{For } V = 0 \qquad & u_\ell = A_\ell \sin(kr - \ell\pi/2) \\ \text{and for } V \neq 0 \qquad & u_\ell = C_\ell \sin(kr - \ell\pi/2 + \delta_\ell) \end{aligned} \quad (16)$$

Thus, switching scattering potential 'on' shifts the phase of the wave function at large distances from the scattering centre.

# Scattering amplitudes from Phase Shifts

Now substituting for $u_\ell$ from Eq.(16) back into Eq.(11) for $\psi(\mathbf{r})$, and after some angular momentum algebra, we obtain a scattering wave function which, when equated with the required asymptotic form of Eq.(10) gives

$$f(\theta) = \frac{1}{k}\sum_\ell (2\ell+1)\; T_\ell \; P_\ell(\cos\theta) \tag{17}$$

$$\text{where} \quad T_\ell = e^{i\delta_\ell}\,\sin\delta_\ell = \frac{1}{2i}(S_\ell - 1)\ . \tag{18}$$

$T_\ell$ is the partial wave $T$-matrix.
$S_\ell$ is the partial-wave $S$-matrix.
They are connected to the **T-matrix** (see later).

There is no dependence on $\varphi$ because of *central* potentials.

# Properties of the S-matrix $S_\ell$:

The S-matrix element is a complex number $S_\ell = e^{2i\delta_\ell}$

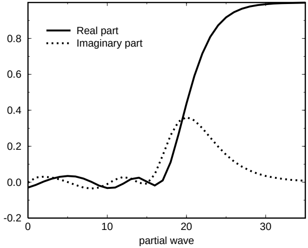
1. For purely diffractive (real) potentials $|S_\ell| = 1$.
   This is called *unitarity*, and is the conservation of flux.
   $\delta_\ell$ is usually positive for attractive potentials.
2. For absorptive (complex) potentials, $|S_\ell| \leq 1$.
   The total absorption=fusion cross section is

$$\sigma_A = \frac{\pi}{k} \sum_\ell (2\ell + 1)(1 - |S_\ell|^2) \qquad (19)$$

# $\ell$-dependence of the S-matrix $S_\ell$:

1. The $\ell$ value (partial wave) where $Re(S_\ell) \sim 0.5$ is the **grazing** $\ell$ value.
2. Partial wave $\ell$ related to impact parameter $b$ in semiclassical limit: $\ell = k\ b$

Typical dependence of $S_\ell$ on $\ell$ for scattering from an absorptive optical potential:

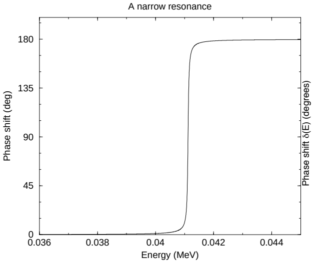


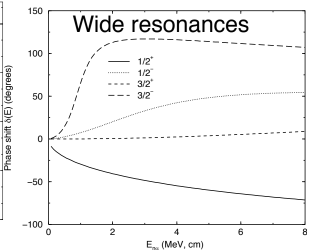

Absorption when $|S_\ell|^2 < 1$ in the interior.

# Resonances:

Occur when two particles trapped together (eg for time $\tau$).

1. Give energy peaks in cross sections with fwhm $\Gamma \sim \hbar/\tau$.
2. Phase shifts typically rise rapidly through $\pi/2$ (90°) as $\delta_{\rm res}(E) = \arctan\left(\frac{\Gamma/2}{E_r - E}\right) + \delta_{bg}$ for peak at energy $E_r$.
3. Cross section peak $\sigma(E) = \frac{4\pi}{k^2}(2L+1)\frac{\Gamma^2/4}{(E-E_r)^2+\Gamma^2/4} + \sigma_{bg}$



# Free Green's function $G_0(E)$

Can write the Schrödinger equation as

$$(E - H)\,\psi \;=\; 0 \qquad \text{or} \qquad (E - H_0)\,\psi \;=\; V\,\psi \tag{20}$$

where $H = H_0 + V$. Thus

$$\psi \;=\; (E - H_0)^{-1}\,V\,\psi \;=\; G_0(E)\,V\,\psi \tag{21}$$

$G_0(E)$ is the **Green's operator**.

Eq.(21) is not general solution for $\psi$ as can add on solution of homogeneous equation

$$(E \;-\; H_0)\,\chi \;=\; 0 \tag{22}$$

# Lipmann-Schwinger equation

General solution of Eq.(20) is

$$\psi \ = \ \chi \ + \ G_0(E)\, V\, \psi \tag{23}$$

This is iterative

$$\psi \ = \ \chi \ + \ G_0\, V\, \chi \ + \ G_0\, V\, G_0\, V\, \chi \ + \ \dots. \tag{24}$$

Eq.(23) can be written in integral form as the **Lipmann-Schwinger** equation

$$\psi(\mathbf{r}) \ = \ \chi(\mathbf{r}) \ + \ \int d\mathbf{r}'\, G(\mathbf{r},\mathbf{r}')\, V(r')\, \psi(\mathbf{r}') \tag{25}$$

where $G(\mathbf{r},\mathbf{r}')$ is the **Green's function**.

# Integral expression for the Scattering Amplitude

The $\chi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ = incident plane wave,

and we use $\psi_{\mathbf{k}}^{(+)}(\mathbf{r})$ for the scattering wave function.

(i.e. incident momentum **k** and $(+)$ for outgoing waves solution).

Comparing Eq.(25) with required asymptotic form for $\psi$ we see that integral term must tend to

$$f(\theta)\,\frac{e^{ikr}}{r} \qquad \text{as} \;\; |\mathbf{r}| \to \infty \quad . \tag{26}$$

Thus, using properties of Green's function, with $\mathbf{k}'$ at angle $\theta$,

$$f(\theta) \;=\; -\frac{m}{2\pi\hbar^2}\int d\mathbf{r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}}\, V(r)\,\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) \quad . \tag{27}$$

# Transition matrix element

In Dirac (bra-ket) notation we write this

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \langle \mathbf{k}' \mid V \mid \psi_{\mathbf{k}}^{(+)} \rangle \quad (28)$$

$$= -\frac{m}{2\pi\hbar^2} T(\mathbf{k}', \mathbf{k}) \quad . \quad (29)$$

$T(\mathbf{k}', \mathbf{k})$ is known as the **Transition matrix element**.

# Reactions Theory II

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# Non-elastic Cross Sections

## Non-elastic Cross Sections

## How?

### A Code and a Book

Non-elastic Cross Sections

How?

A Code and a Book

Physics of Nuclear Reactions

Elastic Scattering

Inelastic Scattering

Transfer Reactions

Breakup Reactions

Fusion Reactions

Compound Nucleus Decays After Fusion

# Multi-channel Scattering

Use for inelastic, transfer, breakup channels (etc)
in addition to elastic.

Two channel (1=elastic, 2=reaction) make coupled channels:

$$\begin{aligned}[T_1 + U_1 - E_1]\psi_1(\mathbf{r}) + V_{12}\psi_2(\mathbf{r}) &= 0 \\ [T_2 + U_2 - E_2]\psi_2(\mathbf{r}) + V_{21}\psi_1(\mathbf{r}) &= 0. \end{aligned} \quad (1)$$

Forward coupling:
$V_{21}\psi_1(\mathbf{r})$ gives effect of channel 1 on channel 2,
Back coupling:
$V_{12}\psi_2(\mathbf{r})$ gives effect of channel 2 on channel 1

# Simplified Multi-channel Scattering gives DWBA

If channel 2 is weak, we can neglect the $V_{12}\psi_2(\mathbf{r})$ term: the back effect on channel 1. This equals the Born Approximation:

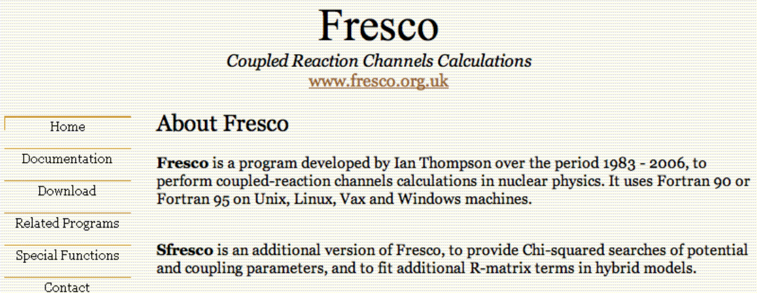
$$[T_1 + U_1 - E_1]\psi_1(\mathbf{r}) + V_{12}\psi_2(\mathbf{r}) = 0$$
$$\psi_2(\mathbf{r}) = -[T_2 + U_2 - E_2]^{-1} V_{21}\psi_1(\mathbf{r}) \quad (2)$$

So the DWBA scattering amplitude in channel 2 is

$$f_{21}(\theta) = -\frac{m_2}{2\pi\hbar^2} \langle \mathbf{k}_2 \mid V_{21} \mid \psi_1 \rangle \quad (3)$$

DWBA is a simplified method to give scattering amplitudes of non-elastic channels.

# Coupled Channels Calculations




Free!

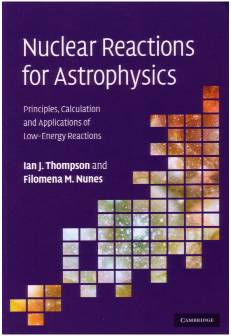
# New Book (available now!)

Nuclear Reactions for Astrophysics

Principles, Calculation and Applications of Low-Energy Reactions

Ian J. Thompson
*Lawrence Livermore National Laboratory, Livermore, CA 94551, U.S.A.*

and Filomena M. Nunes
*National Superconducting Cyclotron Laboratory, East Lansing, MI 48824, U.S.A.*




Cambridge University Press: http://www.cambridge.org/9780521856355

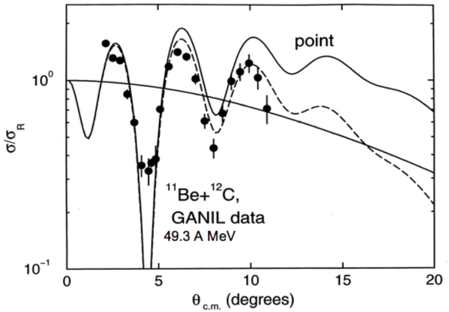
# Physics of Nuclear Reactions

- Halo Scattering: Elastic
- Halo Total Reaction Cross Section
- Transfer Reactions
- Breakup Reactions
- Halo Fusion Reactions

# Halo Scattering: Elastic

Depends on

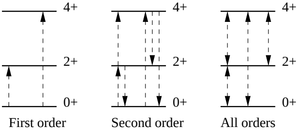
- Folded potential from densities
- Halo breakup effects, i.e.
- Polarisation potential from breakup channel

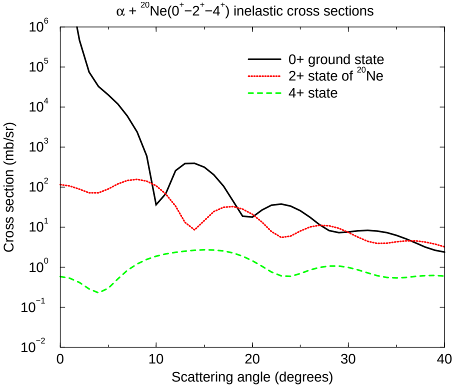


# Inelastic Scattering

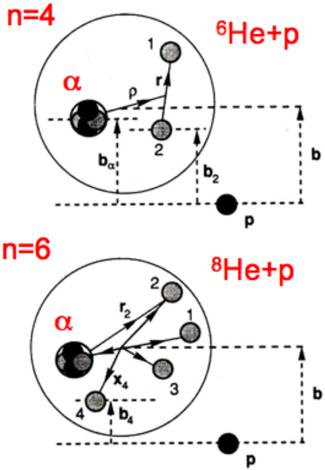
Need a structure model for the couplings
Choose here a rotational model: $\beta_2 = 0.205$.

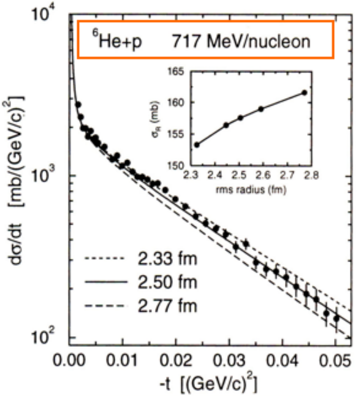
Theory options:



# Four- and Six-body Scattering



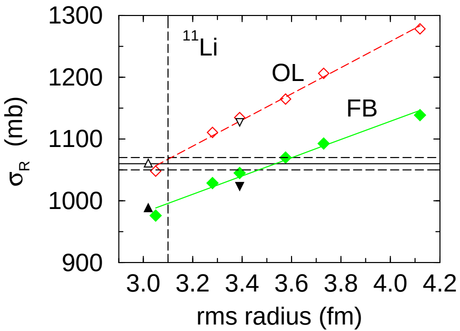




Al-Khalili and Tostevin, PRC **57** (1998) 1846
Tostevin et al., PRC **56** (1997) R2929

# Halo Total Reaction Cross Section

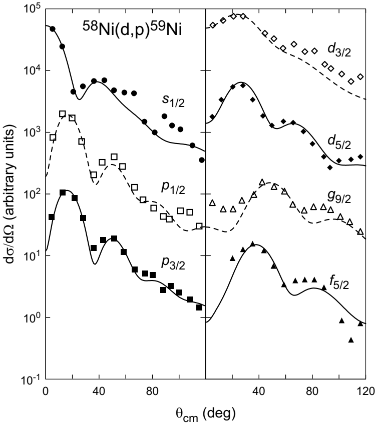
Depends on

- Densities and NN scattering, as usual
- But: Effects of Halo Breakup (virtual and real) are big!
- Use few-body Glauber, not Optical Limit Glauber
- New radii are larger.



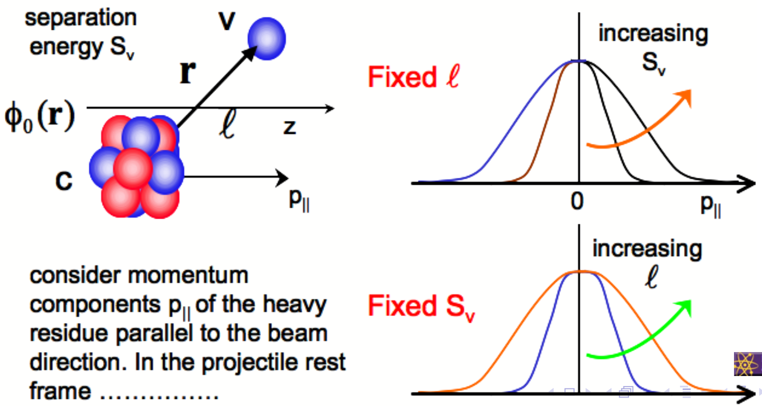
# Transfer Reactions to Probe Single-Particle Structure

- Weak, so use DWBA
- One-nucleon transfers, (p,d)
  shape shows L-value of orbital
  magnitude gives spectroscopic factor
- Two-neutron transfers, (p,t)
  Magnitude depends on s-wave pairing in halo
  Only relative magnitudes reliably modeled.
- But: full analysis requires multi-step calculations



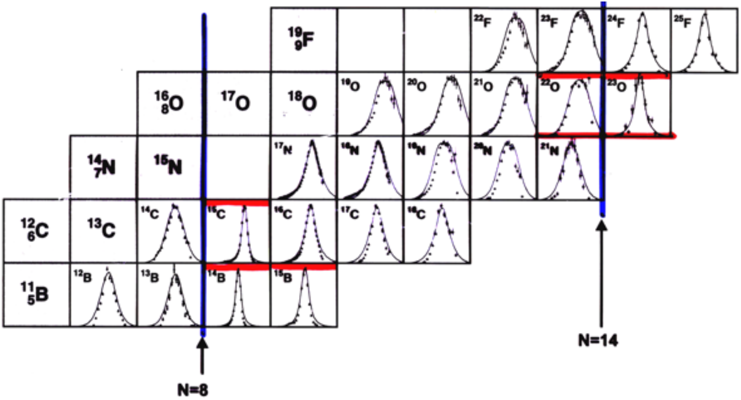
# Stripping (Breakup) Reactions: Measuring Momentum

Probing the momentum content of bound states by breakup reactions



# Stripping Reactions: Nuclear Structure

Glauber (eikonal) theory of breakup:

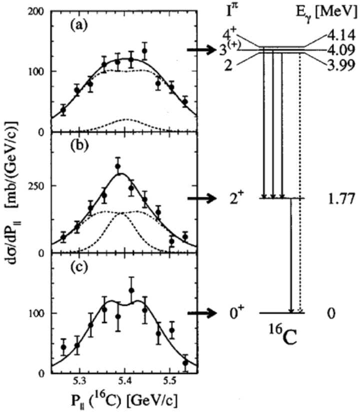



E.Sauvan et al., Phys Lett B 491 (2000) 1

# Stripping Reactions: Removing a Neutron

Reaction $^{9}$Be($^{17}$C, $^{16}$C$\gamma$)X

Measured $\gamma$ from core decays helps to fix the final state




ℓ=0,2 admixture

ℓ=0,2 admixture

pure ℓ=2 but large!!

[Maddalena et al., PRC63(01)024613]

# Halo Fusion: an Unsolved Problem

In low-energy Halo Fusion (near the Coulomb barrier):
Halo neutrons should affect fusion:

- Increase fusion, from neutron attractions & neutron flow
- Decrease complete fusion, from breakup
- Increase fusion, from molecular states & resonances

So: need experiments + good theories!
Some experiments already performed with $^{6}$He and $^{9}$Be, but theoretical interpretations are still unclear.

# Compound Nucleus Decays After Fusion

Flux does not 'disappear' the nuclei fuse together, but reappears as mixture of narrow resonances of the compound system.

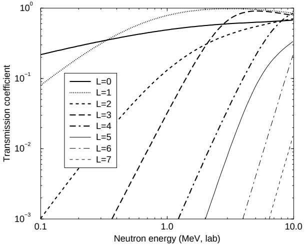
- Narrow resonances $\Rightarrow$ long-lived $\Rightarrow$ many oscillations to decay
- Bohr hypothesis: decay independent of production method
- So decay by all possible means $\alpha$:
  emission of $\gamma$, n, p, $\alpha$, maybe fission.
- Average the cross sections over (say) 0.1 MeV, $\langle\sigma_{\alpha'\alpha}\rangle$ to cover many resonances
- Hauser-Feshbach theory gives the statistical branching ratios between the channels $\alpha$.

So we can calculate residual nuclear ground states after all emissions are finished.

# Transmission coefficients for CN production

'Transmission coefficient' $\mathcal{T}_\alpha(E) = 1 - |S_\alpha(E)|^2$ is the probability of CN production for scattering at energy $E$.

Transmission coefficients for neutrons incident on $^{90}$Zr in various partial waves $L$, using a global optical potential:

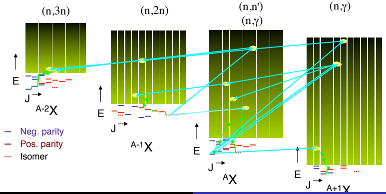


# Decay paths and Branching Probabilities

- So consider all possible exit channels $\alpha''$ and normalize to total
- Hauser-Feshbach cross section $\alpha \to \alpha'$ (simple form):

$$\langle \sigma_{\alpha'\alpha}(L; E) \rangle = \frac{\pi}{k^2}(2L+1)\frac{\mathcal{T}_\alpha \mathcal{T}_{\alpha'}}{\sum_{\alpha''} \mathcal{T}_{\alpha''}}$$

- The same $\mathcal{T}_\alpha$ are used for producing as for decaying.
- If we do not know all the $\alpha$, average over a level density $\rho(E)$:
- Decay paths starting from neutron + $^{A}X$:



# Result of a Hauser-Feshbach Calculation

Using the code TALYS: