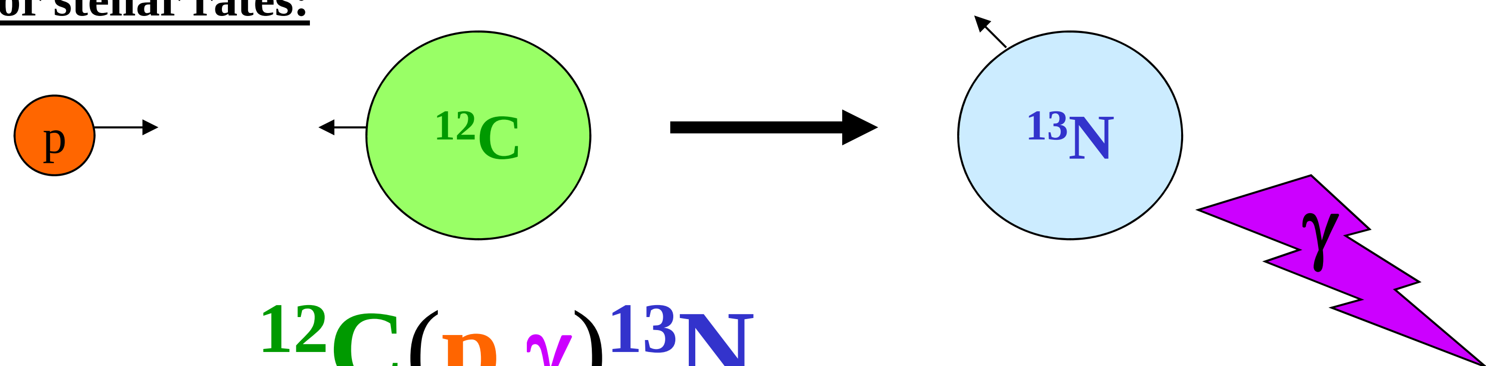
Introduction to stellar reaction rates

Nuclear reactions

- generate energy
- create new isotopes and elements

Notation for stellar rates:



$$^{12}C(p,\gamma)^{13}N$$

The heavier "target" nucleus (Lab: target)

the lighter "incoming projectile" (Lab: beam)

the lighter "outgoing particle" (Lab: residual of beam)

the heavier "residual" nucleus (Lab: residual of target)

(adapted from traditional laboratory experiments with a target and a beam)

Typical reactions in nuclear astrophysics:

$$\left.\begin{array}{l}(\mathrm{p},\mathrm{g})\\(\mathrm{p},\alpha)\\(\mathrm{p},\mathrm{n})\\(\mathrm{n},\gamma)\\(\mathrm{n},\alpha)\end{array}\right\}\quad \text{and their inverses}$$

:

A(B,γ)

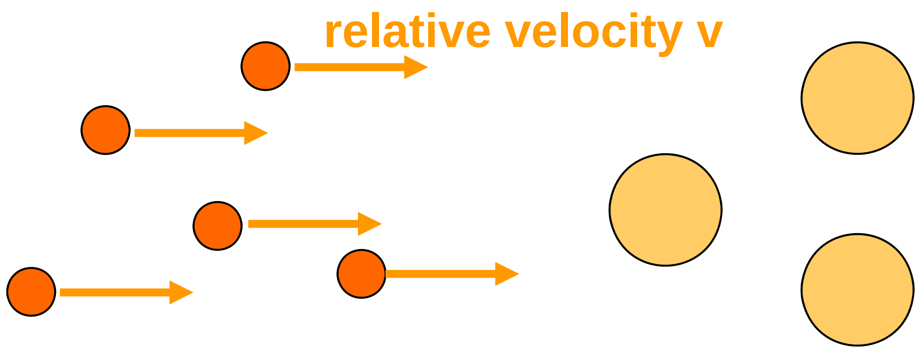
A(B,p)

A(B,α)

A(B,n)

cross section σ

bombard target nuclei with projectiles:



Definition of cross section:

# of reactions per second and target nucleus	=	σ	.	# of incoming projectiles per second and cm^2

or in symbols: $\boxed{\lambda = \sigma j}$

with $\boldsymbol{j}$ as particle number current density.
Of course $\boldsymbol{j = n\,v}$ with particle number density n)

Units for cross section:

1 barn = 10^{-24} cm^2 (= 100 fm^2 or about half the size (cross sectional area) of a uranium nucleus)

Reaction rate in stellar environment

Mix of (fully ionized) projectiles and target nuclei at a temperature T

Reaction rate for relative velocity v

in volume V with projectile number density n_p

$$\lambda = \sigma n_p v$$

$$R = \sigma n_p v n_T V$$ Reactions per second

so for reaction rate per second and cm^3: $$r = n_p n_T \sigma v$$

This is proportional to the number of p-T pairs in the volume.
If projectile and target are identical, one has to divide by 2 to avoid double counting

as there are $\frac{n(n-1)}{2} \approx \frac{n}{2}$ pairs per volume, therefore

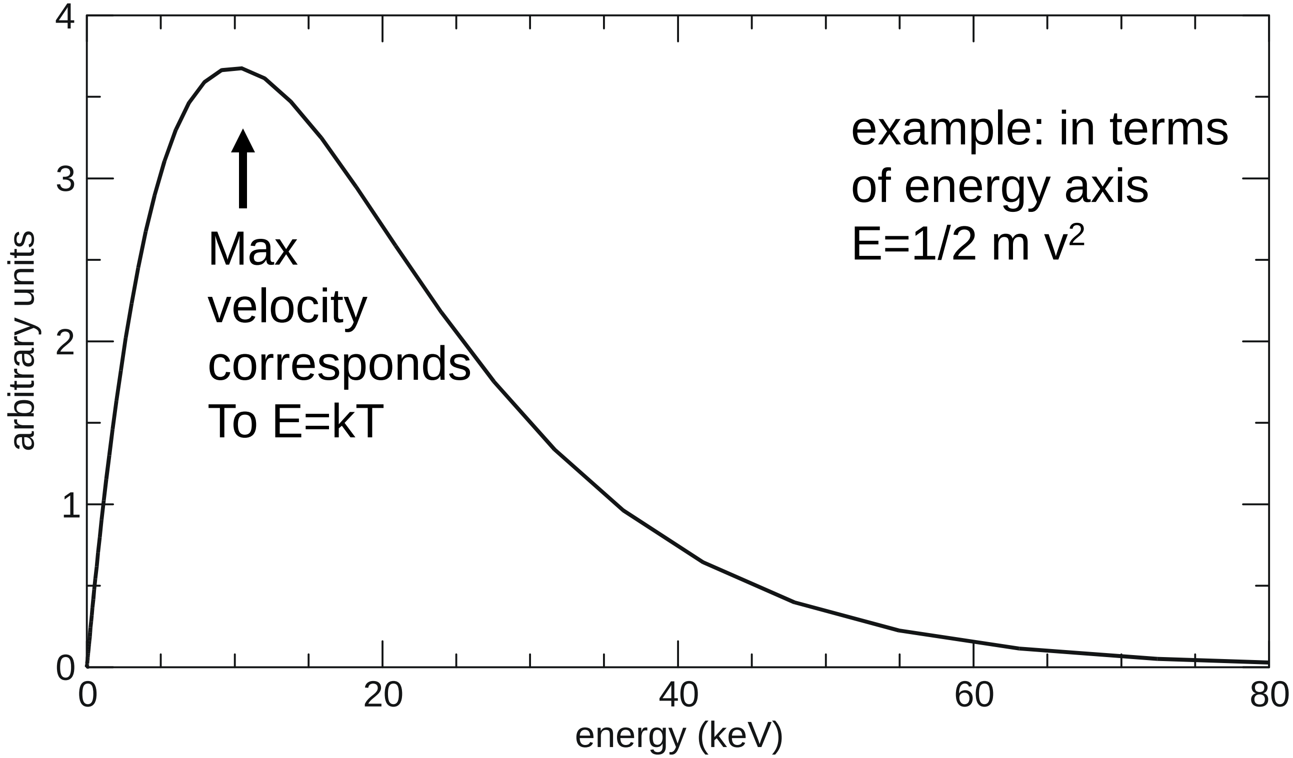
$$\boxed{r = \frac{1}{1+\delta_{pT}} n_p n_T \sigma v}$$

Relative velocities in stars: Maxwell Boltzmann distribution

for most practical applications (for example in stars) projectile and target nuclei are always in thermal equilibrium and follow a Maxwell-Bolzmann velocity distribution:

then the probability $\Phi(v)$ to find a particle with a velocity between v and v+dv is

$$\Phi(v) = 4\pi \left(\frac{m}{2\pi kT} \right)^{3/2} v^2 \, e^{-\frac{m v^2}{2kT}} \qquad \text{with} \qquad \int \Phi(v) dv = 1$$



one can show (Clayton Pg 294-295) that the **relative** velocities between two particles are distributed the same way:

$$\Phi(v) = 4\pi\left(\frac{\mu}{2\pi kT}\right)^{3/2} v^2\, \mathrm{e}^{-\frac{\mu \mathrm{v}^2}{2\mathrm{kT}}}$$

with the mass m replaced by the reduced mass μ of the 2 particle system

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

the stellar reaction rate has to be averaged over the distribution $\Phi(v)$

$$r = \frac{1}{1+\delta_{pT}} n_p n_T \int \sigma(v)\Phi(v) v\,dv$$

typical strong velocity dependence !

or short hand:

$$r = \frac{1}{1+\delta_{pT}} n_p n_T < \sigma v >$$

expressed in terms abundances

$$r = \frac{1}{1+\delta_{pT}} Y_T Y_p \rho^2 \, \mathrm{N}_\mathrm{A}^2 < \sigma v >$$

reactions per s and cm^3

$$\lambda = \frac{1}{1+\delta_{pT}} Y_p \rho \, \underbrace{\mathrm{N}_\mathrm{A} < \sigma v >}$$

reactions per s and target nucleus

this is usually referred to
as the **stellar reaction rate**
of a specific reaction

units of stellar reaction rate N_A<sv>: usually cm^3/s/mole, though in fact
cm^3/s/g would be better (and is needed to verify dimensions of equations)

(Y does not have a unit)

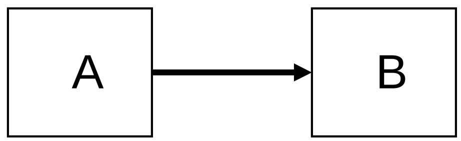
Abundance changes, lifetimes, networks

Lets assume the only reaction that involves nuclei A and B is destruction (production) of A (B) by A capturing the projectile a:

$$A + a \rightarrow B$$

And lets assume the reaction rate is constant over time.

This is a very simple reaction network:



Each isotope is a node that is linked to other isotopes through production and destruction channels

Starting from an initial abundance, we can then ask, how the abundance of each network node evolves over time

Typically the same light projectiles drive most of the reactions (neutron or proton capture) so we don’t enter p, n and all its destruction channels into the graphics but understand that they get produced and destroyed as well)

We can write down a set of differential equations for each abundance change

$$\frac{dn_A}{dt} = -n_A \lambda = -n_A Y_a \rho \, \mathrm{N_A} < \sigma v >$$

$$\frac{dn_B}{dt} = +n_A \lambda$$

Assuming, the reaction rate is constant in time, this case can be solved easily (same as decay law):

$$n_A(t) = n_{0A} \, \mathrm{e}^{-\lambda \mathrm{t}}$$

$$n_B(t) = n_{0A} (1 - \mathrm{e}^{-\lambda \mathrm{t}})$$

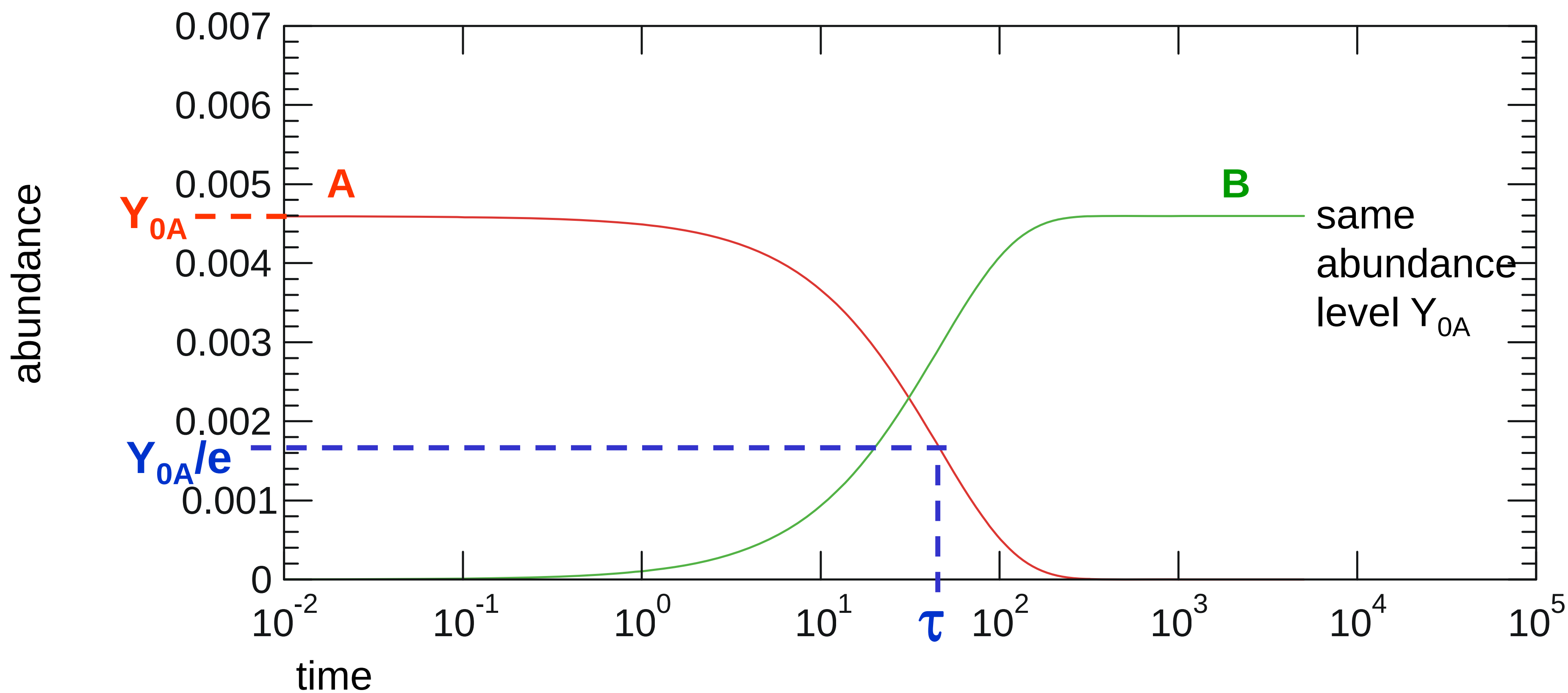
and of course

$$Y_A(t) = Y_{0A}\, e^{-\lambda t}$$

$$Y_B(t) = Y_{0A}(1 - e^{-\lambda t})$$

after some time, nucleus A is entirely converted to nucleus B

Example:



Lifetime of A (against destruction via the reaction A+a) : $\tau = \frac{1}{\lambda} = \frac{1}{Y_a \rho N_A < \sigma v >}$

(of course half-life of A $T_{1/2} = \ln 2/\lambda$)

Energy generation through a specific reaction:

Reaction Q-value: Energy generated (if >0) by a single reaction

in general, for any reaction (sequence) with nuclear masses m:

$$Q = c^2 \left(\sum_{\text{initial nuclei i}} m_i - \sum_{\text{final nuclei j}} m_j \right)$$

Energy generation: Energy generated per g and second by a reaction A+a:

$$\varepsilon = \frac{rQ}{\rho} = Q \frac{1}{1+\delta_{aA}} Y_A Y_a \rho \mathrm{N}_\mathrm{A}^2 < \sigma v >$$

Unit in CGS: erg
(1 erg = 1E-7 Joule)

(remember, positron emission almost always leads to an additional energy release by the subsequent annihilation process (2 x .511 MeV))

Reaction flow

abundance of nuclei of species A converted in time in time interval [t1,t2] into species B via a specific reaction A→B is called reaction flow

$$F_{A\to B} = \int_{t_1}^{t_2} \left(\frac{dY_A}{dt} \right)_{\mathrm{A\to B}} dt = \int_{t_1}^{t_2} \lambda_{A\to B}(t) Y_A(t) dt$$

For **Net reaction flow** subtract the flow via the inverse of that specific reaction (this is what is often plotted in the network connecting the nodes)

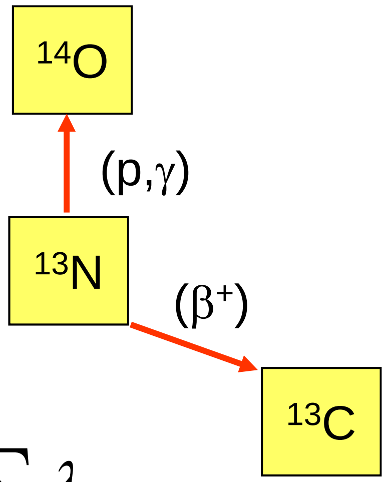
$$F_{\mathrm{net}\ A\to B} = F_{A\to B} - F_{B\to A}$$

(Sometimes the reaction flow is also called reaction flux)

In our example, at infinite time A has been converted entirely into B. Therefore

$$F_{\mathrm{net}\ A\to B}(t\to\infty) = Y_A(t=0)$$

Multiple reactions destroying a nuclide



example: in the CNO cycle, ^{13}N can either capture a proton or β decay.

each destructive reaction i has a rate λ_i

Total lifetime

the total destruction rate for the nucleus is then $$\lambda = \sum_i \lambda_i$$

its total lifetime $$\tau = \frac{1}{\lambda} = \frac{1}{\sum_i \lambda_i}$$

Branching

the reaction flow branching into reaction i, $\boldsymbol{b_i}$ is the fraction of destructive flow through reaction i. (or the fraction of nuclei destroyed via reaction i)

$$b_i = \frac{\lambda_i}{\sum_j \lambda_j}$$

General reaction network

A set of n isotopes with abundances Y_i,

Consider 1- and 2-body rates only

$$\frac{dY_i}{dt} = \boxed{\left[\sum_{j,k} Y_j Y_k \rho N_A < \sigma v >_{jk \to i} + \sum_l \lambda_{l \to i} Y_l \right]}$$

production

$$- \boxed{\left[\sum_m Y_i Y_m \rho N_A < \sigma v >_{im \to \mathrm{any}} + \sum_n \lambda_{i \to n} Y_i \right]}$$

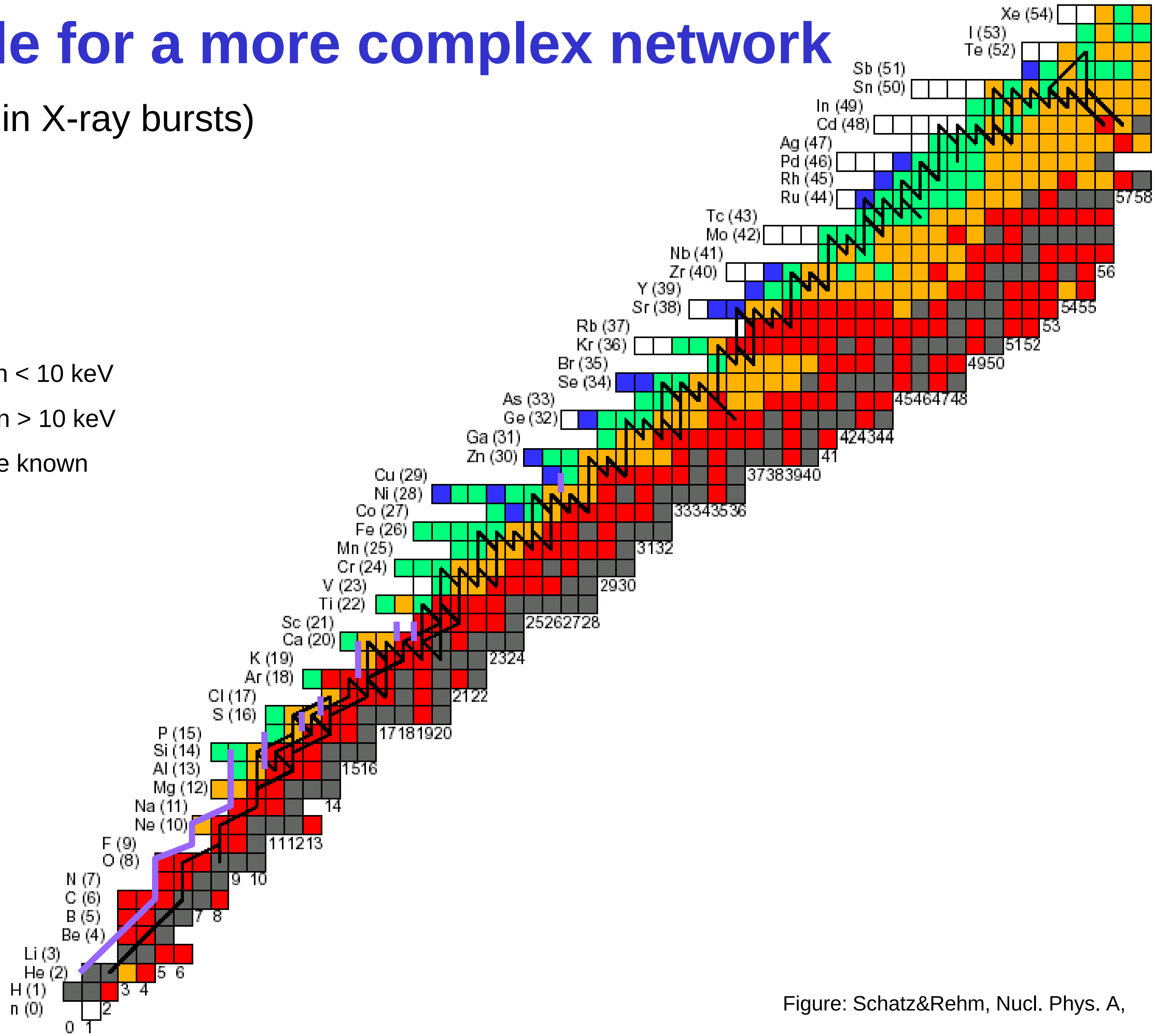
destruction

Note that this depends on mass density ρ and temperature (through $<\sigma v>$ and λ) so this requires input from a stellar model.

Needs to be solved numerically. This is not trivial as system is very stiff (reaction rate timescales vary by many many orders of magnitude)

Example for a more complex network

(rp-process in X-ray bursts)



Mass known < 10 keV

Mass known > 10 keV

Only half-life known

seen

Figure: Schatz&Rehm, Nucl. Phys. A,