Simulation of Beam and Plasma Systems
Homework 6

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Problem 1 - Warp: parallel plates and Child-Langmuir law

Let us assume two parallel conducting plates separated by the distance d, and with a constant potential difference V between the two plates.

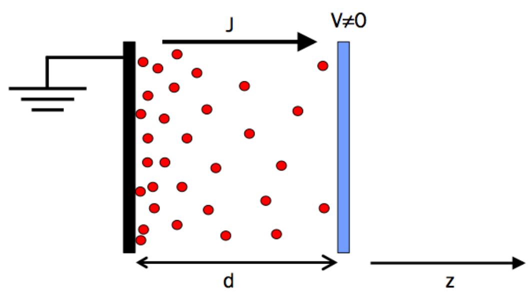

Figure 1: Two parallel conducting plates separated by the distance d and powered by a voltage difference V will, for a proper value of V, accelerate charged particle initially at rest at one of the plates.

Charged particles injected with initial velocity $v = 0$ will be accelerated toward the opposite plate when the sign of the voltage difference is set appropriately.

a) Run a parallel plate example in warp. You can download the script from `https://raw.githubusercontent.com/RemiLehe/uspas_exercise/master/Parallel_plates_injection.py`, as usual.

- Open the input script "Parallel_plates_injection.py".
- Run the script: "python -i Parallel_plates_injection.py".
- Observe the injection of particles in window 0, and of the current, charge density, electric field and potential in window 1. You may run additional steps by using the command "step(n)" where n is the number of time steps to run.
- In the same terminal, after exiting the run, or in another terminal window, open the output file created by window(0) with the command "gist Parallel_plates_injection.000.cgm". At the gist prompt, type 'help' for the help menu. Select the graphic window and type "f" multiple times, followed by "b" multiple times. Refer to the help menu for additional

options. You may also explore the graphics produced in window(1) by opening the file with “gist current.cgm”.

- Look at the input script line-by-line until the end, trying to understand the meaning of each command.
- Search for the line ”diode_current = 1.”. Augment the diode_current up to 6A, by increment of 1, rerunning the script each time. What do you observe?

b) Let us examine why the emitted current does not seem to exceed a certain limit in steady-steady state mode.
Assuming a steady state flow, the current will be constant ($\mathbf{J} = \rho\mathbf{v}$) and the energy of a particle at a given position is given by $1/2mv^2 = -q\phi(z)$ where q, m, v and z are respectively the charge, mass, velocity and position of the particle. Using Poisson's equation ($d^2\phi/dz^2 = -\rho/\epsilon_0$), and posing $\Phi = -q\phi$, show that

$$\frac{d^2\Phi}{dz^2} = \frac{q\mathbf{J}\sqrt{m/2}}{\epsilon_0}\Phi^{-1/2}. \tag{1}$$

Multiply each side by $d\Phi/dz$ and integrate to find

$$\left(\frac{d\Phi}{dz}\right)^2 = \frac{4q\mathbf{J}\sqrt{m/2}}{\epsilon_0}\Phi^{1/2} + C. \tag{2}$$

Considering now the special case $d\Phi/dz = 0$ at $z = 0$, we get $C = 0$, integrate again to find

$$\frac{4}{3}\Phi^{3/4} = 2\sqrt{\frac{q\mathbf{J}}{\epsilon_0}}\left(\frac{m}{2}\right)^{1/4} z. \tag{3}$$

Using $\phi(d) = V$, find that

$$\mathbf{J} = \frac{4}{9}\epsilon_0\sqrt{\frac{2}{m}}\frac{(-qV)^{3/2}}{qd^2}, \tag{4}$$

and

$$\phi(z) = V\left(\frac{z}{d}\right)^{4/3}. \tag{5}$$

The expression

$$\mathbf{J} = \frac{4}{9}\epsilon_0\sqrt{\frac{2|q|}{m}}\frac{|V|^{3/2}}{d^2}, \tag{6}$$

is known as the *Child's Law* or *Child-Langmuir Law* and gives the maximum current that can be extracted for a given voltage and plate separation.

c) Compare the theoretical Child-Langmuir limit with the steady state values in warp.

- Find the command “plzprofiles(l_CL=False)” and replace by “plzprofiles(l_CL=True)”.
- The input script will now fail at execution. Fix the input script by replacing the expressions for Child-Langmuir Law and the potential dependency at the appropriate locations.
- At the end of the simulation, the simulation printed ”Maximum current from Child-Langmuir law = ***”. Replace the input value for the current by this value, restart the simulation and verify that the emitted current, charge density, electric field and potential profiles converge toward the ones predicted at the Child-Langmuir limit when steady-state is reached.

Problem 2 - Elegant: Bunch Compression

Preamble: Open the Sirepo/elegant simulation you created during the Monday afternoon computer lab and go to 'Lattice'. You should see a linac section followed by a compression chicane, constructed of four dipoles, which is then followed by a FODO cell with four quadrupoles.

You will primarily look at four diagnostics in the Visualization tab:

- `run_setup.output`: longitudinal phase space of the bunch at the end of the beam line
- `hist_chicane`: a histogram taken at the entrance to the chicane
- `hist_chicane_end`: a histogram taken at the exit of the chicane
- `hist_final`: a histogram at the very end of the beam line

a) Before the simulation shows bunch compression, you need to calculate the optimal accelerating phase ϕ in the linac. The compression ratio is given by:

$$C = \frac{1}{1 + hR_{56}}$$

where h is the linear chirp and R_{56} comes from the R-matrix for the lattice.

To calculate the linear chirp you will use:

$$h = \frac{kV_{tot}\cos\phi}{E_0 + V_{tot}\sin\phi}$$

Here, k is the wavenumber of the linac cavities, V_{tot} is the linac voltage contribution from all cavities, E_0 is the initial energy of the beam in eV, and ϕ is the accelerating phase.

Document all of the following steps in writing:

1. There are 86 cells in each cavity section and 4 total cavity sections in the beam line. You can find the voltage for a single cell by finding the RFCA element 'R1' and checking the number for voltage. Use this to calculate the total accelerating voltage of the linac.
2. Look at the RFCA element 'R1' to find the cavity frequency. Use this to calculate k.
3. Go to the source tab and look at the value of 'Central momentum of the beamline' this is the initial momentum of the bunch at the linac entrance. Use this calculate the energy E_0.
4. Go to Visualization and run the simulation. Look at matrix_output.SDDS_output and plot R_{56} vs s to find R_{56} at the end of the chicane.
5. Now that you have all the components, calculate the optimal phase ϕ to maximize the compression ratio C. When you have this number verify it with the instructor. Then in lattice got to RPN variables (under Beamline Elements) and put ϕ in the value for 'PHASE'.

b) Go to 'Visualization' and run the simulation.

1. Look at plots of tFrequency vs. t in hist_chicane and hist_chicane_end. What has happened to the longitudinal distribution of the bunch?

2. Now change the histograms to plot deltaFrequency vs. delta. Do the distributions look different at the beginning and end of the chicane?
3. In run_setup.output make sure p vs. t is plotted. Save a picture of the longitudinal phase space. Comment on what you see.

Problem 3 - Alternative particle pusher

In order to integrate the continuous equations of motion

$$\frac{d\boldsymbol{x}}{dt} = \frac{\boldsymbol{p}}{\gamma m} \qquad \frac{d\boldsymbol{p}}{dt} = q\left(\boldsymbol{E} + \frac{\boldsymbol{p}}{\gamma m} \times \boldsymbol{B}\right) \quad (\text{with } \gamma = \sqrt{1+\boldsymbol{p}^2/(mc)^2})$$

we choose to use the following staggered scheme

$$\frac{\boldsymbol{x}^{n+1} - \boldsymbol{x}^n}{\Delta t} = \frac{\boldsymbol{p}^{n+1/2}}{\gamma^{n+1/2} m} \tag{7}$$

$$\frac{\boldsymbol{p}^{n+1/2} - \boldsymbol{p}^{n-1/2}}{\Delta t} = q\left(\boldsymbol{E}^n + \frac{1}{2}\left(\frac{\boldsymbol{p}^{n+1/2}}{\gamma^{n+1/2} m} + \frac{\boldsymbol{p}^{n-1/2}}{\gamma^{n-1/2} m}\right) \times \boldsymbol{B}^n\right) \tag{8}$$

where the superscripts ^{n}, $^{n+1/2}$ and $^{n+1}$ indicates that the corresponding quantity is taken at time $n\Delta t$, $(n+1/2)\Delta t$ and $(n+1)\Delta t$, and where $\gamma^{n+1/2} = \sqrt{1+(\boldsymbol{p}^{n+1/2})^2/(mc)^2}$ and $\gamma^{n-1/2} = \sqrt{1+(\boldsymbol{p}^{n-1/2})^2/(mc)^2}$.

While equation (7) is easy to convert into an update equation ($\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \frac{\boldsymbol{p}^{n+1/2}}{\gamma^{n+1/2} m}\Delta t$), it is more difficult to obtain $\boldsymbol{p}^{n+1/2}$ from $\boldsymbol{p}^{n-1/2}$, since equation (8) is implicit (i.e. it involves $\boldsymbol{p}^{n+1/2}$ in both its right-hand side and left-hand side). The aim of this problem is to obtain the corresponding update equation.

a) Verify that the following scheme

$$\boldsymbol{p}' = \boldsymbol{p}^{n-1/2} + q\boldsymbol{E}^n \Delta t + \boldsymbol{p}^{n-1/2} \times \boldsymbol{s} \qquad \boldsymbol{s} = \frac{q\Delta t \boldsymbol{B}^n}{2\gamma^{n-1/2} m} \tag{9}$$

$$\boldsymbol{p}^{n+1/2} = \boldsymbol{p}' + \boldsymbol{p}^{n+1/2} \times \boldsymbol{t} \qquad \boldsymbol{t} = \frac{q\Delta t \boldsymbol{B}^n}{2\gamma^{n+1/2} m} \tag{10}$$

satisfies equation (8). Which one of the two above equations is implicit? **Note: The two equations are consecutive steps in the algorithm.**

b) By taking the vector product and scalar product of (10) by $\boldsymbol{t}$, show that $\boldsymbol{p}^{n+1/2}$ can be extracted from this equation and reads

$$\boldsymbol{p}^{n+1/2} = \frac{1}{1+\boldsymbol{t}^2}\left(\boldsymbol{p}' + \boldsymbol{p}' \times \boldsymbol{t} + (\boldsymbol{p}' \cdot \boldsymbol{t})\boldsymbol{t}\right) \tag{11}$$

Reminder: For any set of 3 vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, one has $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$

c) In fact, equation (11) is still not explicit, since the expression of $\boldsymbol{t}$ (in equation (7)) depends on $\boldsymbol{p}^{n+1/2}$, through $\gamma^{n+1/2}$.

Thus, in order to extract $\boldsymbol{p}^{n+1/2}$ explicitly from equation (11), take the following steps:

- From equation (11), show that the expression of $(\boldsymbol{p}^{n+1/2})^2$ is:

$$(\boldsymbol{p}^{n+1/2})^2 = \frac{(\boldsymbol{p}')^2 + (\boldsymbol{p}' \cdot \boldsymbol{t})^2}{1 + \boldsymbol{t}^2} \tag{12}$$

  Reminder: For any set of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, one has $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}^2 \boldsymbol{b}^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$

- Using the notation

$$\boldsymbol{\tau} = \frac{q\boldsymbol{B}\Delta t}{2m} = \gamma^{n+1/2}\boldsymbol{t} \tag{13}$$

  show that equation (12) leads to

$$(\gamma^2)^2 + [\tau^2 - (\gamma')^2]\gamma^2 - [(u)^2 + \tau^2] = 0 \tag{14}$$

  where γ is a short-hand notation for $\gamma^{n+1/2}$ (in other words $\gamma^2 = 1 + (\boldsymbol{p}^{n+1/2})^2/(mc)^2$) and where the following notations where used

$$\tau^2 = \boldsymbol{\tau}^2 \qquad u = \frac{\boldsymbol{p}' \cdot \boldsymbol{\tau}}{mc} \qquad (\gamma')^2 = 1 + \frac{(\boldsymbol{p}')^2}{(mc)^2} \tag{15}$$

- By remarking that equation (14) is a second-order polynomial in γ^2, show that its solution is:

$$\gamma = \frac{1}{\sqrt{2}}\sqrt{(\gamma')^2 - \tau^2 + \sqrt{[(\gamma')^2 - \tau^2]^2 + 4u^2 + 4\tau^2}} \tag{16}$$

d) Summing up all the previous step, in an actual algorithm (e.g. in Python) that computes $\boldsymbol{p}^{n+1/2}$ from $\boldsymbol{p}^{n-1/2}$, in **which order** should the following steps be executed?

   i) Compute $\gamma^{n+1/2}$ (also denoted here as γ) using equation (16) and equation (15).

   ii) Compute $\boldsymbol{s}$ using equation (7), and $\boldsymbol{\tau}$ using equation (13)

   iii) Compute $\boldsymbol{p}^{n+1/2}$ using equation (11)

   iv) Compute $\boldsymbol{p}'$ from $\boldsymbol{p}^{n-1/2}$ using equation (9)

   v) Compute $\boldsymbol{t}$ from $\boldsymbol{\tau}$ and $\gamma^{n+1/2}$ (using equation (13))

e) Download the script `particle_pusher.py` from

   `https://raw.githubusercontent.com/RemiLehe/uspas_exercise/master/particle_pusher.py`

   This is an incomplete script that implements the solver considered here. Find the lines tagged by `## ASSIGNEMENT` and complete them with the appropriate code.

f) Run the script (`python particle_pusher.py`). It integrates the equations of motion for a relativistic electron ($\gamma \simeq 100$), in a magnetic field of 1 Tesla.

   What do you expect the motion to be? Are the results from the code quantitatively consistent with the expected Larmor period of $\tau = \frac{2\pi\gamma m_e}{eB_0}$ (with $B_0 = 1\ T$, $m_e = 0.9 \times 10^{-30}\ kg$ and $e = 1.6 \times 10^{-19}\ C$) ?