Electromagnetic wave propagation in Particle-In-Cell codes

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Electromagnetic Waves: Outline

1D discrete propagation equation in vacuum

Reminder: 1D discrete Maxwell equations in vacuum

$$\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n-1/2}_{\ell+1/2}}{\Delta t} = -\frac{{E_x}^{n}_{\ell+1} - {E_x}^{n}_{\ell}}{\Delta z} \quad \text{(from } \partial_t \boldsymbol{B} = -\boldsymbol{\nabla}\times\boldsymbol{E}\text{)}$$

$$\frac{1}{c^2}\frac{{E_x}^{n+1}_{\ell} - {E_x}^{n}_{\ell}}{\Delta t} = -\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n+1/2}_{\ell-1/2}}{\Delta z} \quad \text{(from } \frac{1}{c^2}\partial_t \boldsymbol{E} = \boldsymbol{\nabla}\times\boldsymbol{B}\text{)}$$

These equations can be combined into a **propagation equation** for E_x:

$$\frac{1}{c^2}\frac{{E_x}^{n+1}_{\ell} - {E_x}^{n}_{\ell}}{\Delta t^2} - \frac{1}{c^2}\frac{{E_x}^{n}_{\ell} - {E_x}^{n-1}_{\ell}}{\Delta t^2} = -\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n+1/2}_{\ell-1/2}}{\Delta z\Delta t} + \frac{{B_y}^{n-1/2}_{\ell+1/2} - {B_y}^{n-1/2}_{\ell-1/2}}{\Delta z\Delta t}$$

$$= -\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n-1/2}_{\ell+1/2}}{\Delta z\Delta t} + \frac{{B_y}^{n+1/2}_{\ell-1/2} - {B_y}^{n-1/2}_{\ell-1/2}}{\Delta z\Delta t}$$

$$= \frac{{E_x}^{n}_{\ell+1} - {E_x}^{n}_{\ell}}{\Delta z^2} - \frac{{E_x}^{n}_{\ell} - {E_x}^{n}_{\ell-1}}{\Delta z^2}$$

1D discrete propagation equation in vacuum

$$\frac{1}{c^2}\frac{{E_x}^{n+1}_{\ell} - 2{E_x}^{n}_{\ell} + {E_x}^{n-1}_{\ell}}{\Delta t^2} = \frac{{E_x}^{n}_{\ell+1} - 2{E_x}^{n}_{\ell} + {E_x}^{n}_{\ell-1}}{\Delta z^2} \qquad i.e. \ \frac{1}{c^2}\partial_t^2 E_x|^n_\ell = \partial_z^2 E_x|^n_\ell$$

1D dispersion relation

1D discrete propagation equation in vacuum

$$\frac{1}{c^2}\frac{{E_x}^{n+1}_{\ell} - 2{E_x}^{n}_{\ell} + {E_x}^{n-1}_{\ell}}{\Delta t^2} = \frac{{E_x}^{n}_{\ell+1} - 2{E_x}^{n}_{\ell} + {E_x}^{n}_{\ell-1}}{\Delta z^2}$$

→ **Von Neumann analysis:** assume the solutions of this equation are of the form $E_0 e^{ikz - i\omega t}$ (propagating wave), i.e.

$${E_x}^{n}_{\ell} = E_0\, e^{ik\,\ell\Delta z - i\omega n\Delta t}$$

Replacing this ansatz into the discrete progagation equation yields

$$\frac{e^{ik\ell\Delta z}}{c^2}\frac{e^{-i\omega(n+1)\Delta t} - 2e^{-i\omega n\Delta t} + e^{-i\omega(n-1)\Delta t}}{\Delta t^2} = e^{-i\omega n\Delta t}\frac{e^{ik(\ell+1)\Delta z} - 2e^{ik\ell\Delta z} + e^{ik(\ell-1)\Delta z}}{\Delta z^2}$$

$$\frac{e^{ik\ell\Delta z - i\omega n\Delta t}}{c^2}\frac{e^{-i\omega\Delta t} - 2 + e^{i\omega\Delta t}}{\Delta t^2} = e^{ik\ell\Delta z - i\omega n\Delta t}\frac{e^{ik\Delta z} - 2 + e^{-ik\Delta z}}{\Delta z^2}$$

$$\frac{1}{c^2}\frac{(e^{-i\omega\Delta t/2} - e^{i\omega\Delta t/2})^2}{\Delta t^2} = \frac{(e^{ik\Delta z/2} - e^{-ik\Delta z/2})^2}{\Delta z^2}$$

1D dispersion relation

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{1}{\Delta z^2}\sin^2\left(\frac{k\Delta z}{2}\right) \qquad \text{(instead of } \omega^2 = c^2k^2\text{)}$$

$c\Delta t \leq \Delta z$ $\rightarrow$ Numerical dispersion

For $c\Delta t \leq \Delta z$, the discrete dispersion relation

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{1}{\Delta z^2}\sin^2\left(\frac{k\Delta z}{2}\right)$$

has real solutions ω, for any k:

$$\omega = \pm\frac{2}{\Delta t}\arcsin\left(\frac{c\Delta t}{\Delta z}\sin\left(\frac{k\Delta z}{2}\right)\right)$$

Thus, the phase velocity $v_\phi = \omega/k$ is:

$$v_\phi = \pm\frac{2}{k\Delta t}\arcsin\left(\frac{c\Delta t}{\Delta z}\sin\left(\frac{k\Delta z}{2}\right)\right)$$

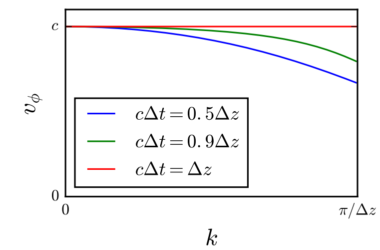
Numerical dispersion

In a PIC code, the **electromagnetic waves** propagate (in vacuum) at a **velocity which depends on** k (and on Δt, Δz),

instead of propagating at the speed of light: $v_\phi = \pm c$

$c\Delta t \leq \Delta z$ $\rightarrow$ Numerical dispersion

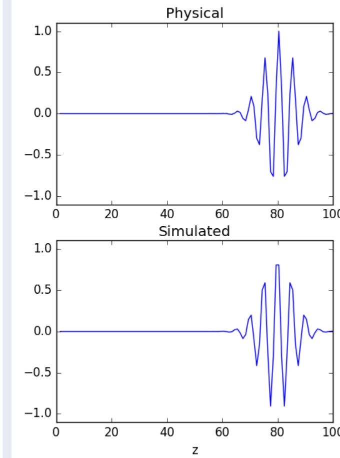
$$v_\phi = \frac{2}{k\Delta t}\arcsin\left(\frac{c\Delta t}{\Delta z}\sin\left(\frac{k\Delta z}{2}\right)\right)$$



NB: $k = \pi/\Delta z$, $\lambda = 2\Delta z$: shortest wavelength supported by the grid.

The shorter the wavelength, the slower the propagation.

Animation: $c\Delta t = 0.5\Delta z$



$c\Delta t > \Delta z$ $\rightarrow$ Courant limit

For $c\Delta t > \Delta z$, the discrete dispersion relation

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{1}{\Delta z^2}\sin^2\left(\frac{k\Delta z}{2}\right)$$

has **no real solutions** ω, for k close to $\pi/\Delta z$. The solution ω is **imaginary** and the corresponding mode is **unstable**.

Courant limit (a.k.a. CFL limit)

Standard EM-PIC codes are **unstable** for $c\Delta t > \Delta z$ (in 1D).

- Thus, practical use of **electromagnetic PIC** codes is restricted to $\Delta t \leq \Delta z/c$.
- For a given spatial resolution Δz, this limits **how fast** a simulation can advance in time.
- **Electrostatic PIC codes** do not have this limitation
$\rightarrow$ Can be much faster than EM-PIC codes to simulate a system over a given period of time, by taking **large timesteps** Δt.

Electromagnetic Waves: Outline

Dispersion and Courant limit in 3D

Derivation of dispersion relation

Combine discrete Maxwell equation → Discrete propagation equation → Von Neumann analysis → Numerical dispersion relation

Same process in 3D. The Von Neumann analysis assumes:

$$E = E_0 e^{ik_x x + ik_y y + ik_z z - i\omega t}$$

3D Numerical dispersion relation

$$\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{c^2\Delta t^2} = \frac{\sin^2\left(\frac{k_x\Delta x}{2}\right)}{\Delta x^2} + \frac{\sin^2\left(\frac{k_y\Delta y}{2}\right)}{\Delta y^2} + \frac{\sin^2\left(\frac{k_z\Delta z}{2}\right)}{\Delta z^2}$$

instead of the physical dispersion $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2)$

Courant limit (a.k.a CFL limit) in 3D

$$c\Delta t \leq \frac{1}{\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}$$

Numerical dispersion in 3D

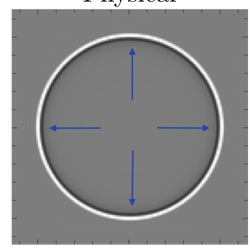
3D Discrete dispersion relation

$$\frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{c^2\Delta t^2} = \frac{\sin^2\left(\frac{k_x\Delta x}{2}\right)}{\Delta x^2} + \frac{\sin^2\left(\frac{k_y\Delta y}{2}\right)}{\Delta y^2} + \frac{\sin^2\left(\frac{k_z\Delta z}{2}\right)}{\Delta z^2}$$

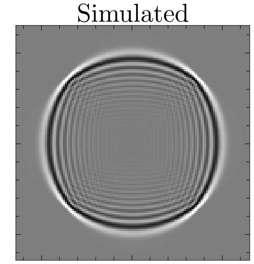
Velocity depends on the **wavelength and propagation direction**.

Example: expanding electromagnetic wave

Physical



Simulated

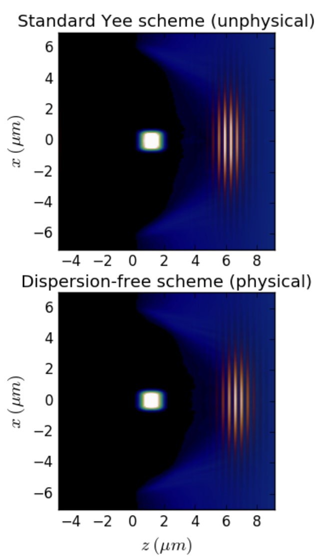


Even for $\Delta t = \Delta t_{CFL}$: waves are **slower than** c along the main axes.

Impact of numerical dispersion

Animation: laser-wakefield acceleration

- A short and intense **laser pulse**, followed by a relativistic **electron bunch**, enters a **plasma** (generated from a gas jet).
- The laser pulse generates a **wake** in the plasma, with **electric fields** that can **accelerate** the electron bunch.
- Simulation with the Yee scheme (and low resolution):
  - The laser is **artificially slow** (numerical dispersion)
  - Thus the electron bunch **catches up** with the laser very soon!



Electromagnetic Waves: Outline

1. Numerical dispersion and Courant limit
   - Dispersion and Courant limit in 1D
   - Dispersion and Courant limit in 3D
   - Spectral solvers and numerical dispersion
2. Open boundaries conditions
   - Silver-Müller boundary conditions
   - Perfectly Matched Layers

Yee scheme

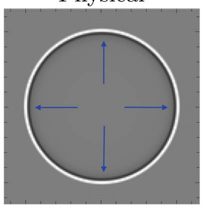
Finite-difference in space and time

e.g. continuous equation : $$\frac{\partial B_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

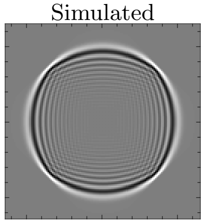
$\rightarrow$ discrete equation : $$B_z^{n+1/2} = B_z{}^{n-1/2} - \Delta t(\hat{\partial}_x E_y|^n - \hat{\partial}_y E_x|^n)$$

with $\hat{\partial}_x F|^n_{i,j,\ell} = \frac{F^n_{i+\frac{1}{2},j,\ell} - F^n_{i-\frac{1}{2},j,\ell}}{\Delta x}$

Physical



Simulated



- Anisotropic
- Waves propagate **slower** than c.

Pseudo-spectral solver

Fourier transform in space, finite-difference in time

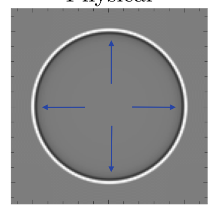
e.g. continuous equation : $$\frac{\partial B_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$\rightarrow$ Fourier space : $$\frac{\partial \hat{\mathcal{B}}_z}{\partial t} = -\left(ik_x\hat{\mathcal{E}}_y - ik_y\hat{\mathcal{E}}_x\right)$$

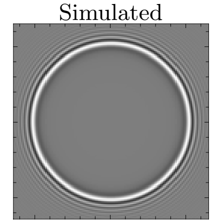
$\rightarrow$ Finite difference in time : $$\hat{\mathcal{B}}_z^{n+1/2} = \hat{\mathcal{B}}_z^{n-1/2} - \Delta t\left(ik_x\hat{\mathcal{E}}_y^n - ik_y\hat{\mathcal{E}}_x^n\right)$$

$\rightarrow$ Use backwards FFT to obtain $B_z^{n+1/2}$ from $\hat{\mathcal{B}}_z^{n+1/2}$

Physical



Simulated



- Isotropic
- Waves propagate **faster** than c.

Analytical pseudo-spectral solver (Haber et al., 1973)

Fourier transform in space, finite-difference in time

e.g. continuous equation : $$\frac{\partial B_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

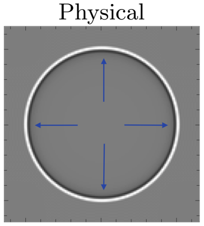
$\rightarrow$ Fourier space : $$\frac{\partial \hat{\mathcal{B}}_z}{\partial t} = -\left(ik_x\hat{\mathcal{E}}_y - ik_y\hat{\mathcal{E}}_x\right)$$

$\rightarrow$ Analytical integration of the coupled Maxwell equations in time:

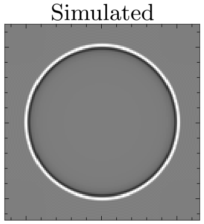
$$\hat{\mathcal{B}}_z^{n+1} = \cos(kc\Delta t)\hat{\mathcal{B}}_z^n - \frac{\sin(kc\Delta t)}{kc}\left(ik_x\hat{\mathcal{E}}_y^n - ik_y\hat{\mathcal{E}}_x^n\right) \qquad k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$\rightarrow$ Use backwards FFT to obtain B_z^{n+1} from $\hat{\mathcal{B}}_z^{n+1}$

Physical



Simulated



- Isotropic
- Waves propagate **exactly** at c.

Dispersion and Courant limit: conclusions

- Electromagnetic solvers have a **maximum value** for the timestep Δt (Courant limit), which depends on the dimension (and the method of discretization)
- Below the Courant limit, waves may propagate at speeds that **artificially differ** from c (numerical dipersion).
  This can have a strong impact in some physical situations.
- Spectral solvers can mitigate (or even eliminate) numerical dispersion.

Electromagnetic Waves: Outline

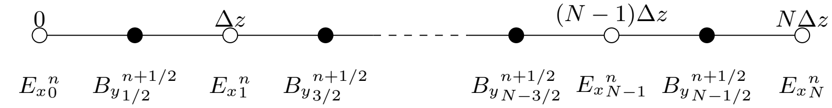
1. Numerical dispersion and Courant limit
   - Dispersion and Courant limit in 1D
   - Dispersion and Courant limit in 3D
   - Spectral solvers and numerical dispersion
2. Open boundaries conditions
   - Silver-Müller boundary conditions
   - Perfectly Matched Layers

Boundary conditions and EM-PIC

Reminder: 1D discrete Maxwell equations in vacuum

$$\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell+1/2}^{n-1/2}}{\Delta t} = -\frac{{E_x}_{\ell+1}^{n} - {E_x}_{\ell}^{n}}{\Delta z}$$

$$\frac{{E_x}_{\ell}^{n+1} - {E_x}_{\ell}^{n}}{\Delta t} = -c^2\frac{{B_y}_{\ell+1/2}^{n+1/2} - {B_y}_{\ell-1/2}^{n+1/2}}{\Delta z}$$

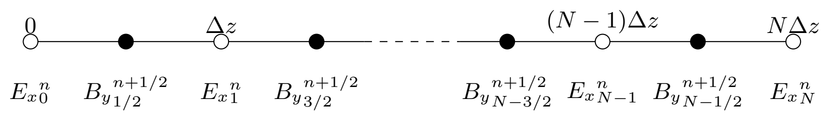


The grid is finite:

- For $\ell = 0$: ${B_y}_{\ell-1/2}^{n+1/2}$ is undefined.
- For $\ell = N$: ${B_y}_{\ell+1/2}^{n+1/2}$ is undefined.

$\rightarrow$ **Assumptions** are needed, for the value of ${B_y}_{-1/2}^{n+1/2}$ and ${B_y}_{N+1/2}^{n+1/2}$.

Boundary conditions and EM-PIC



Typical assumptions

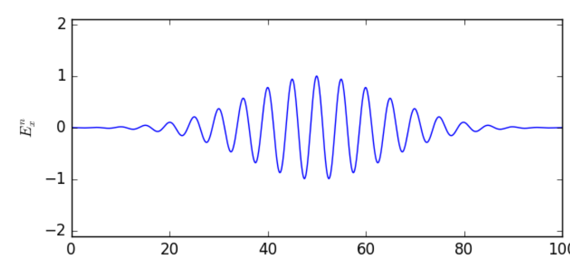
- Periodic: ${B_y}_{-1/2}^{n+1/2} = {B_y}_{N-1/2}^{n+1/2}$ and ${B_y}_{N+1/2}^{n+1/2} = {B_y}_{1/2}^{n+1/2}$
- Dirichlet: ${B_y}_{-1/2}^{n+1/2} = 0$ and ${B_y}_{N+1/2}^{n+1/2} = 0$
- Neumann: ${B_y}_{-1/2}^{n+1/2} = {B_y}_{1/2}^{n+1/2}$ and ${B_y}_{N+1/2}^{n+1/2} = {B_y}_{N-1/2}^{n+1/2}$
  (i.e. $\partial_z B_y|_0^{n+1/2} = 0$ and $\partial_z B_y|_N^{n+1/2} = 0$)

Boundary conditions and EM-PIC

Problem:

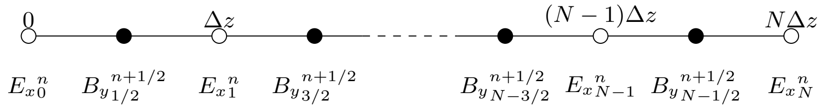
Dirichlet and Neumann boundary conditions **reflect** the EM waves.
For many physical problems, we need the boundaries to **absorb** the waves.

Animation: Neumann boundary conditions



This is because, physically, an **outgoing wave** does not satisfy $B_y(n\Delta z) = 0$ (Dirichlet) or $\partial_z B_y(n\Delta z) = 0$ (Neumann)

Silver-Müller absorbing boundary (right-hand side)



The value of $B_{y\,N+1/2}^{\ n+1/2}$ should be chosen so as to be **consistent with an outgoing wave.**

Physically, for an outgoing wave propagating to the right (from Maxwell's equation):

$$B_y(z,t) = \frac{1}{c}E_x(z,t)$$

Numerically, we can express it as:

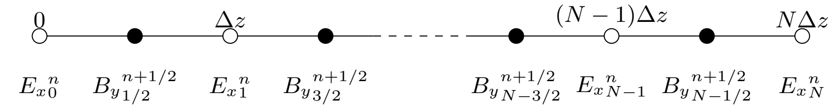
$$B_y|_N^{n+1/2} = \frac{1}{c}E_x|_N^{n+1/2}$$

Because of **staggering**:

$$\frac{B_{y\,N+1/2}^{\ n+1/2} + B_{y\,N-1/2}^{\ n+1/2}}{2} = \frac{1}{c}\frac{E_{x\,N}^{\ n+1} + E_{x\,N}^{\ n}}{2}$$

Numerical dispersion and Courant limit | Open boundaries conditions | References

Silver-Müller absorbing boundary (right-hand side)



By combining the equations:

$$\frac{B_{y\,N+1/2}^{\ n+1/2} + B_{y\,N-1/2}^{\ n+1/2}}{2} = \frac{1}{c}\frac{E_{x\,N}^{\ n+1} + E_{x\,N}^{\ n}}{2} \quad \text{(right-propagating wave)}$$

$$\frac{E_{x\,N}^{\ n+1} - E_{x\,N}^{\ n}}{\Delta t} = -c^2\frac{B_{y\,N+1/2}^{\ n+1/2} - B_{y\,N-1/2}^{\ n+1/2}}{\Delta z} \quad \text{(Maxwell equation)}$$

we obtain

Silver-Müller boundary condition (right-hand side)

$$E_{x\,N}^{\ n+1} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right)E_{x\,N}^{\ n} + \frac{2c^2\Delta t}{c\Delta t + \Delta z}B_{y\,N-1/2}^{\ n+1/2}$$

See e.g. Bjorn Engquist (1977)

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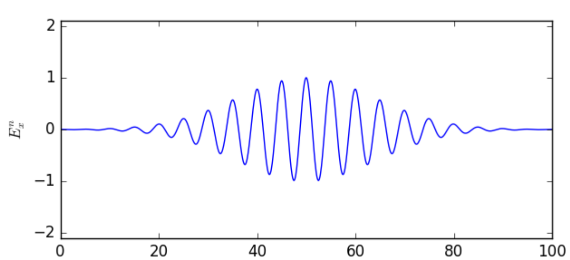
Silver-Müller absorbing boundary (right-hand side)

Silver-Müller boundary condition (right-hand side)

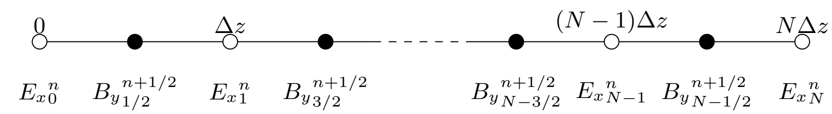
$$E_{x\,N}^{\ n+1} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right)E_{x\,N}^{\ n} + \frac{2c^2\Delta t}{c\Delta t + \Delta z}B_{y\,N-1/2}^{\ n+1/2}$$

Animation: Silver-Müller boundary conditions



Numerical dispersion and Courant limit | Open boundaries conditions | References

Silver-Müller absorbing boundary (left-hand side)



By combining the equations:

$$\frac{B_{y1/2}^{\ n+1/2} + B_{y-1/2}^{\ n+1/2}}{2} = -\frac{1}{c}\frac{E_{x0}^{\ n+1} + E_{x0}^{\ n}}{2} \quad \text{(left-propagating wave)}$$

$$\frac{E_{x0}^{\ n+1} - E_{x0}^{\ n}}{\Delta t} = -c^2\frac{B_{y1/2}^{\ n+1/2} - B_{y-1/2}^{\ n+1/2}}{\Delta z} \quad \text{(Maxwell equation)}$$

we obtain

Silver-Müller boundary condition (left-hand side)

$$E_{x0}^{\ n+1} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right)E_{x0}^{\ n} - \frac{2c^2\Delta t}{c\Delta t + \Delta z}B_{y1/2}^{\ n+1/2}$$

Silver-Müller absorbing boundary in 3D

Maxwell equation:

$$\frac{E_x{}^{n+1}_{i+\frac{1}{2},j,\ell} - E_x{}^{n}_{i+\frac{1}{2},j,\ell}}{c^2\Delta t} = \frac{B_z{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},0} - B_z{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},0}}{\Delta y} - \frac{B_y{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j,\ell+\frac{1}{2}} - B_y{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j,\ell-\frac{1}{2}}}{\Delta z}$$

Silver-Müller boundary condition (left-hand side)

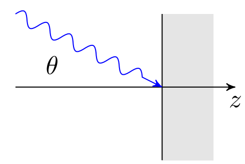
$$E_x{}^{n+1}_{i+\frac{1}{2},j,0} = \left(1 - \frac{2c\Delta t}{c\Delta t + \Delta z}\right) E_x{}^{n}_{i+\frac{1}{2},j,0} - \frac{2c^2\Delta t}{c\Delta t + \Delta z} B_y{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j,\frac{1}{2}}$$
$$+ c^2\Delta t \frac{B_z{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j+\frac{1}{2},0} - B_z{}^{n+\frac{1}{2}}_{i+\frac{1}{2},j-\frac{1}{2},0}}{\Delta y}$$

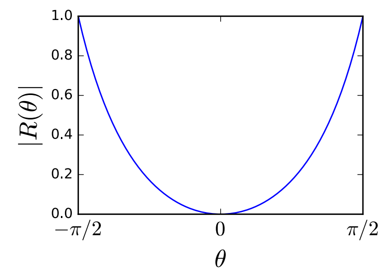
+ Similar equations for the right-hand side
+ Similar equations for B_x and E_y

Silver-Müller absorbing boundary in 3D

Limitation

In 3D, the Silver-Müller boundary conditions are only well-adapted for waves in **normal incidence**.
The reflection coefficient $R(\theta)$ quickly increases with the angle of incidence θ.





Electromagnetic Waves: Outline

Perfectly Matched Layers (in 2D)

Perfectly Matched Layers (Berenger, 1994)

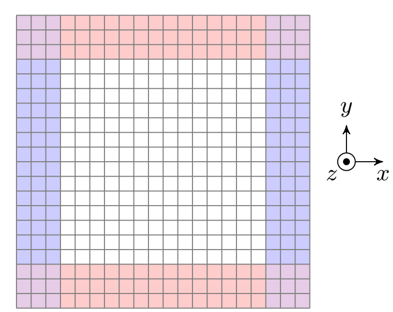
Surround the simulation box by **additional layers of cells**, where the Maxwell equations are **modified** so as to **progressively damp** the waves.

In the bulk:

$$\partial_t E_x = c^2 \partial_y B_z$$
$$\partial_t E_y = -c^2 \partial_x B_z$$
$$\partial_t B_z = -\partial_x E_y + \partial_y E_x$$

In e.g. the right-hand layer:

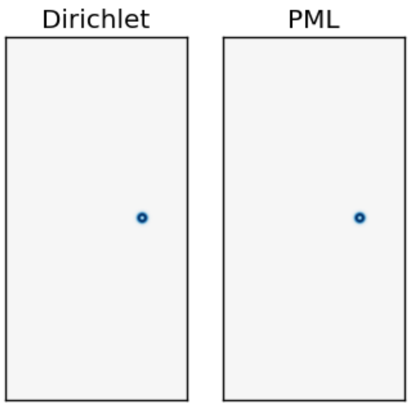
$$\partial_t E_x = c^2 \partial_y B_z$$
$$\partial_t E_y = -c^2 \partial_x B_z - \frac{\boldsymbol{\sigma}}{\boldsymbol{\epsilon_0}} \boldsymbol{E_y}$$
$$B_z = B_{zx} + B_{zy}$$
$$\partial_t B_{zx} = -\partial_x E_y - \frac{\boldsymbol{\sigma}}{\boldsymbol{\epsilon_0}} \boldsymbol{B_{zx}}$$
$$\partial_t B_{zy} = \partial_y E_x$$



Modified Maxwell equations:

- Artificial (unphysical) conductivity σ
- The B_z field is (artificially) split in two

Perfectly Matched Layers (in 2D)

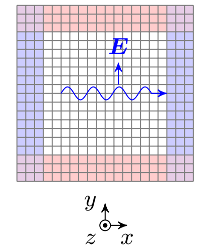


Animation with propagating waves:

- Waves in normal incidence are **absorbed**.
- Waves in grazing incidence **propagate** as if they did not "feel" the boundary.

Perfectly Matched Layers: normal incidence

Explanation based on **continuous equations**



Transverse EM wave propagating along x

$$E_x = 0 \quad E_y \neq 0 \quad \to B_{zy} = 0 \quad B_z = B_{zx}$$

In the bulk:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x \end{aligned} \quad \to \quad \begin{aligned} \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y \end{aligned}$$

In the right-hand layer:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ B_z &= B_{zx} + B_{zy} \\ \partial_t B_{zx} &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_{zx} \\ \partial_t B_{zy} &= \partial_y E_x \end{aligned} \quad \to \quad \begin{aligned} \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ \partial_t B_z &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_z \end{aligned}$$

Perfectly Matched Layers: normal incidence

There is a solution (continuous in E_y and B_z) with **no reflected wave**.

In the bulk ($x < 0$):

$$\begin{aligned} \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y \end{aligned}$$

Solution:

$$E_y = E_0 \cos(k(x - ct))$$

$$B_z = \frac{E_0}{c} \cos(k(x - ct))$$

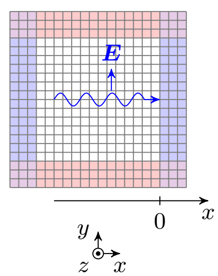
In the right-hand layer ($x > 0$):

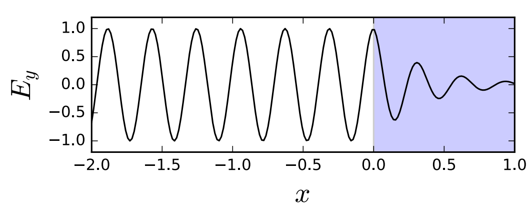
$$\begin{aligned} \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ \partial_t B_z &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_z \end{aligned}$$

Solution:

$$E_y = E_0 \cos(k(x - ct)) e^{-\frac{\sigma}{\epsilon_0 c} x}$$

$$B_z = \frac{E_0}{c} \cos(k(x - ct)) e^{-\frac{\sigma}{\epsilon_0 c} x}$$



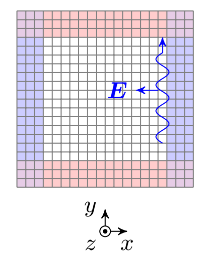


The wave is damped before reaching the end of the outer layer.

Perfectly Matched Layers: grazing incidence

Transverse EM wave propagating along y

$$E_x \neq 0 \quad E_y = 0 \quad \to B_{zx} = 0 \quad B_z = B_{zy}$$



In the bulk:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z \\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x \end{aligned} \quad \to \quad \begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t B_z &= \partial_y E_x \end{aligned}$$

In the right-hand layer:

$$\begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t E_y &= -c^2 \partial_x B_z - \frac{\sigma}{\epsilon_0} E_y \\ B_z &= B_{zx} + B_{zy} \\ \partial_t B_{zx} &= -\partial_x E_y - \frac{\sigma}{\epsilon_0} B_{zx} \\ \partial_t B_{zy} &= \partial_y E_x \end{aligned} \quad \to \quad \begin{aligned} \partial_t E_x &= c^2 \partial_y B_z \\ \partial_t B_z &= \partial_y E_x \end{aligned}$$

The propagation equations are **identical** in the bulk and the outer layer.
A wave in **grazing indidence** does not "feel" the boundary.

Open boundary conditions: conclusion

- If no **special care** is taken at the boundary, it will **by default** produce a reflected wave.
- **Silver-Müller boundary conditions:**
  - Easy to implement
  - But only cancels reflection for waves at normal incidence
- **Perfectly Matched Layers**
  - Need extra layers of cells, where the Maxwell equations are artificially modified.
  - The anisotropic Maxwell equations lead to proper behavior for waves with any incidence angle.

References

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