Simulation of Beam and Plasma Systems
Homework 9 - Final Exam

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Problem 1 - Maxwell's equations and redundant information.

a) Show that the relativistic Vlasov equation

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + q\left[\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right]\cdot\frac{\partial}{\partial \boldsymbol{p}}\right\} f(\boldsymbol{x},\boldsymbol{p},t) = 0$$

with

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{\gamma m} = \frac{\boldsymbol{p}/m}{\left[1 + \boldsymbol{p}^2/(mc)^2\right]^{1/2}}$$

implies conservation of charge with

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \boldsymbol{x}}\cdot\mathbf{J} = 0$$

where

$$\begin{aligned}\rho &= q\int d^3p\, f \\ \mathbf{J} &= q\int d^3p\, \boldsymbol{v} f\end{aligned}$$

Hint 1: Use the same steps as in the previous homework problem.

Hint 2: You are permitted to do this non-relativisticly ($\gamma = 1$) if you want. The result holds either way.

b) The 3D Maxwell equations are linear, 1^{st}-order-in-time, kinematic equations for $\boldsymbol{E}(\boldsymbol{x},t)$, $\boldsymbol{B}(\boldsymbol{x},t)$ if $\rho(\boldsymbol{x},t)$ and $\mathbf{J}(\boldsymbol{x},t)$ are regarded as prescribed sources.

1) How many field components are in $\boldsymbol{E}$, $\boldsymbol{B}$?

2) How many equations are in the standard set of Maxwell equations?

$$\begin{aligned}\nabla\cdot\boldsymbol{E} &= \frac{\rho}{\epsilon_0} \\ \nabla\times\boldsymbol{E} &= -\frac{\partial}{\partial t}\boldsymbol{B} \\ \nabla\cdot\boldsymbol{B} &= 0 \\ \nabla\times\boldsymbol{B} &= \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\boldsymbol{E}\end{aligned}$$

3) What initial ($t = 0$) values are required for $\boldsymbol{E}$ and $\boldsymbol{B}$ to solve the Maxwell equations for $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ for all times t with $\rho(\boldsymbol{x}, t)$ and $\mathbf{J}(\boldsymbol{x}, t)$ specified?

c) To better understand the situation in b), show that the Maxwell equations imply that

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{E}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} &= -\mu_0 \frac{\partial}{\partial t}\mathbf{J} \\ \nabla \times (\nabla \times \boldsymbol{B}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{B} &= \mu_0 \nabla \times \mathbf{J} \end{aligned}$$

where

$$\mu_0 \epsilon_0 = \frac{1}{c^2}.$$

$\rightarrow$ 6 2^{nd}-order-in-time equations for 6 field components $\boldsymbol{E}$, $\boldsymbol{B}$.

d) Show that

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$$

implies that

$$\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{B} = 0.$$

Does this imply that $\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0$ for all times t if $\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t = 0) = 0$?

e) Show that

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\boldsymbol{E} \qquad \text{and} \qquad \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \boldsymbol{x}} \cdot \mathbf{J} = 0$$

imply that

$$\frac{\partial}{\partial t}\left(\nabla \cdot \boldsymbol{E} - \rho/\epsilon_0\right) = 0.$$

Does this imply that $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ for all times t if $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ at $t = 0$?

f) Show that the Maxwell equations

$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\boldsymbol{E} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

imply that

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \boldsymbol{x}} \cdot \mathbf{J} = 0.$$

Does this imply that the Maxwell equations should only be applied to charge ρ and current sources $\mathbf{J}$ which locally conserve charge?

g) Based on a) - f), can we solve Maxwell's equations for $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ for sources ρ and $\mathbf{J}$ satisfying

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \boldsymbol{x}} \cdot \mathbf{J} = 0$$

for all times $t \geq 0$ by only solving the two Maxwell equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B} \qquad \text{and} \qquad \nabla \times \boldsymbol{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\boldsymbol{E}$$

provided that

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B} = 0$$

are satisfied at time $t = 0$? If yes, does it matter if we use $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$ and $\nabla \cdot \boldsymbol{B} = 0$? Why?

Problem 2 - Extended-stencil Maxwell solver

Let us consider the following scheme for the 1D Maxwell equations in vacuum

$$\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n-1/2}_{\ell+1/2}}{\Delta t} = -\left(\frac{{E_x}^{n}_{\ell+1} - {E_x}^{n}_{\ell}}{\Delta z}\right)$$

$$\frac{{E_x}^{n+1}_{\ell} - {E_x}^{n}_{\ell}}{\Delta t} = -c^2\left((1-\alpha)\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n+1/2}_{\ell-1/2}}{\Delta z} + \alpha\frac{{B_y}^{n+1/2}_{\ell+3/2} - {B_y}^{n+1/2}_{\ell-3/2}}{3\Delta z}\right)$$

where ${B_y}^{n'}_{\ell'}$ and ${E_x}^{n'}_{\ell'}$ represent the fields B_y and E_x at time $n'\Delta t$ and position $\ell'\Delta z$.

a) By performing a Taylor expansion of the B field to order 2 in Δz (i.e. with an error term $O(\Delta z^3)$), show that the expression

$$(1-\alpha)\frac{{B_y}^{n+1/2}_{\ell+1/2} - {B_y}^{n+1/2}_{\ell-1/2}}{\Delta z} + \alpha\frac{{B_y}^{n+1/2}_{\ell+3/2} - {B_y}^{n+1/2}_{\ell-3/2}}{3\Delta z}$$

Is an approximation of $\frac{\partial B}{\partial z}$ at position $\ell\Delta z$ (and time $(n+1/2)\Delta t$) which is accurate to order 2 in Δz.

Hint: Remember that ${B_y}^{n+1/2}_{\ell+1/2}$ denotes the B_y field at position $z = (\ell+1/2)\Delta z$. In other words: ${B_y}^{n+1/2}_{\ell+1/2} = {B_y}^{n+1/2}(\ell\Delta z + \Delta z/2) = {B_y}^{n+1/2}(\ell\Delta z) + \left.\frac{\partial B_y^{n+1/2}}{\partial z}\right|_{\ell\Delta z}\left(\frac{\Delta z}{2}\right) + \frac{1}{2!}\left.\frac{\partial^2 B_y^{n+1/2}}{\partial z^2}\right|_{\ell\Delta z}\left(\frac{\Delta z}{2}\right)^2 + \mathcal{O}(\Delta z^3)$ (with similar expressions for ${B_y}^{n+1/2}_{\ell-1/2}$, ${B_y}^{n+1/2}_{\ell+3/2}$, ${B_y}^{n+1/2}_{\ell-3/2}$).

b) By combining the discrete Maxwell equations, show that the corresponding discrete propagation equation for E_x is of the form:

$$\begin{aligned}\frac{{E_x}^{n+1}_{\ell} - 2{E_x}^{n}_{\ell} + {E_x}^{n-1}_{\ell}}{c^2\Delta t^2} &= \frac{{E_x}^{n}_{\ell+1} - 2{E_x}^{n}_{\ell} + {E_x}^{n}_{\ell-1}}{\Delta z^2} \\ &+ \beta\frac{{E_x}^{n}_{\ell+2} - 4{E_x}^{n}_{\ell+1} + 6{E_x}^{n}_{\ell} - 4{E_x}^{n}_{\ell-1} + {E_x}^{n}_{\ell-2}}{\Delta z^2}\end{aligned} \qquad (1)$$

and give the expression of β as a function of α.

Hint: Start by evaluating the quantity

$$\frac{1}{\Delta t}\left(\frac{{E_x}^{n+1}_{\ell} - {E_x}^{n}_{\ell}}{\Delta t} - \frac{{E_x}^{n}_{\ell} - {E_x}^{n-1}_{\ell}}{\Delta t}\right)$$

using the second Maxwell equation from above.

c) By assuming that E_x is of the form

$${E_x}^{n}_{\ell} = E_0 e^{ik\ell\Delta z - i\omega n\Delta t}$$

Show that the discrete dispersion relation is:

$$\frac{1}{c^2\Delta t^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{1}{\Delta z^2}\sin^2\left(\frac{k\Delta z}{2}\right) - \frac{4\alpha}{3\Delta z^2}\sin^4\left(\frac{k\Delta z}{2}\right)$$

d) For $\alpha < 3/8$, show that the right-hand side of the above equation is a growing function of k, for $k \in [0, \pi/\Delta z]$.

Infer the maximum value that the right-hand takes for $k \in [0, \pi/\Delta z]$, and thus infer that the Courant limit is

$$\Delta t_{CFL} = \frac{\Delta z}{c} \frac{1}{\sqrt{1 - \frac{4\alpha}{3}}}$$

Reminder for standard formulas:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Problem 3 - Python ODE solver

Write a python script to advance the moments

$$\frac{d}{ds} \begin{bmatrix} \langle \tilde{x}^2 \rangle_\perp \\ \langle \tilde{x}\tilde{x}' \rangle_\perp \\ \langle \tilde{x}'^2 \rangle_\perp \\ \langle \tilde{y}^2 \rangle_\perp \\ \langle \tilde{y}\tilde{y}' \rangle_\perp \\ \langle \tilde{y}'^2 \rangle_\perp \end{bmatrix} = \begin{bmatrix} 2\langle \tilde{x}\tilde{x}' \rangle_\perp \\ \langle \tilde{x}'^2 \rangle_\perp - \kappa_x(s)\langle \tilde{x}^2 \rangle_\perp + \frac{Q\langle \tilde{x}^2 \rangle_\perp^{1/2}}{2[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \\ -2\kappa_x(s)\langle \tilde{x}\tilde{x}' \rangle_\perp + \frac{Q\langle \tilde{x}\tilde{x}' \rangle_\perp}{\langle \tilde{x}^2 \rangle_\perp^{1/2}[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \\ 2\langle \tilde{y}\tilde{y}' \rangle_\perp \\ \langle \tilde{y}'^2 \rangle_\perp - \kappa_y(s)\langle \tilde{y}^2 \rangle_\perp + \frac{Q\langle \tilde{y}^2 \rangle_\perp^{1/2}}{2[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \\ -2\kappa_y(s)\langle \tilde{y}\tilde{y}' \rangle_\perp + \frac{Q\langle \tilde{y}\tilde{y}' \rangle_\perp}{\langle \tilde{y}^2 \rangle_\perp^{1/2}[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \end{bmatrix}$$

with $\kappa_x(s) = -\kappa_y(s) = \hat{\kappa} \cos\left(\frac{2\pi s}{L_p}\right)$. Use a perveance of $\mathrm{Q} = 6 \times 10^{-4}$, focus strength $\hat{\kappa} = \frac{30}{\text{meter}^2}$, and lattice period $\mathrm{L_p} = 0.5$ m.

a) Use a scientific python package with an ODE integrator to evolve the second order moments from an (arbitrary) initial condition at $s = 0$.

Hint: You can use `scipy.integrate.odeint`.

(import it by typing `from scipy.integrate import odeint`) In ipython you can type `odeint?` for more information. The default tolerance of this function is not sufficient to resolve the problem, so you will need to pass the argument `atol=1e-11` or lower to get consistent results. Ask for help if you are stuck!

b) Apply the result in a) to advance from the initial condition

$$\begin{aligned} \langle \tilde{x}^2 \rangle_\perp = \langle \tilde{y}^2 \rangle_\perp &= \frac{1}{4} \text{ mm}^2 \\ \langle \tilde{x}\tilde{x}' \rangle_\perp = \langle \tilde{y}\tilde{y}' \rangle_\perp &= 0 \\ \langle \tilde{x}'^2 \rangle_\perp = \langle \tilde{y}'^2 \rangle_\perp &= 25 \text{ mrad}^2 \end{aligned}$$

Use `matplotlib` to plot all 2^{nd} order moments on an axial mesh with at least 10 grid points over 10 lattice periods.

c) Plot the combination of moments corresponding to rms x-emittance

$$\epsilon_{x,rms} = \sqrt{\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}'^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2} \quad [mm\text{-}mrad]$$

vs. s for 10 periods. Should you expect this value to be constant to numerical precision?

d) Show that initial conditions at $s = 0$ can be found for some values of $\langle \tilde{x}^2 \rangle_\perp$ and $\langle \tilde{y}^2 \rangle_\perp$ with initial conditions

$$\begin{aligned} \langle \tilde{x}\tilde{x}' \rangle_\perp = \langle \tilde{y}\tilde{y}' \rangle_\perp &= 0 \\ \langle \tilde{x}'^2 \rangle_\perp = \langle \tilde{y}'^2 \rangle_\perp &= 25 \text{ mrad}^2 \end{aligned}$$

such that all moments repeat to numeral precision at $s = \mathrm{L_p}$. Is $\epsilon_{x,rms}$ still constant?

Hint: Try numerical root finding from initial guess values of $\langle \tilde{x}^2 \rangle_\perp = \langle \tilde{y}^2 \rangle_\perp$. You can either write your own short routine or use `fsolve` from `scipy.optimize`. In the latter case, you need to pass the argument `factor=1.` to keep the function from crashing. You may need to investigate setting the tolerance as well (similar to a).

Problem 4 - Sirepo/elegant

a) Create a copy of an existing Sirepo/elegant simulation, by pasting this URL into your browser: `https://uspas-sirepo.radiasoft.org/elegant#/source/o7oYeBDe`

1 Modify the 4^{th} dipole of your chicane by enabling the `OUTPUT FILE` parameter, on page 5 of the parameter input window.

- Make sure that `N Kicks = 16` for the dipole.
- Make sure you have steady-state CSR turned on by setting `value = 1` for the `alter_elements` command with `item = STEADY_STATE, name = BEND?`.

2 Go to the Visualization tab and click "Start new simulation".

- You may see an error message: "elegant Errors: warning: 7 elements had no matrix", but you can ignore it.
- In the window for `BEND4`, plot `DeltaGamma` vs. `s`.
- This is a plot of the CSR wakefield along the bunch with the field plotted in units of $\Delta\gamma/m$.
- Each plot is at one of the 16 steps through the dipole.
- Rewind the movie to the beginning, then step through the images one by one.
- Observe how the wake evolves as the beam enters the dipole.
- Go back to the beginning again, then step one by one through the first 5 images, saving each of them to a file.
- The first image should look like Figure 1.

3 Go back to commands and turn on transient CSR effects (`value = 0` for the `alter_elements` command with `item = STEADY_STATE`, `name = BEND?`).

- Now run the simulation and again look at `BEND4` plotting `s` vs. `DeltaGamma`.
- Compare how the wakefield at the beginning of the dipole now appears.

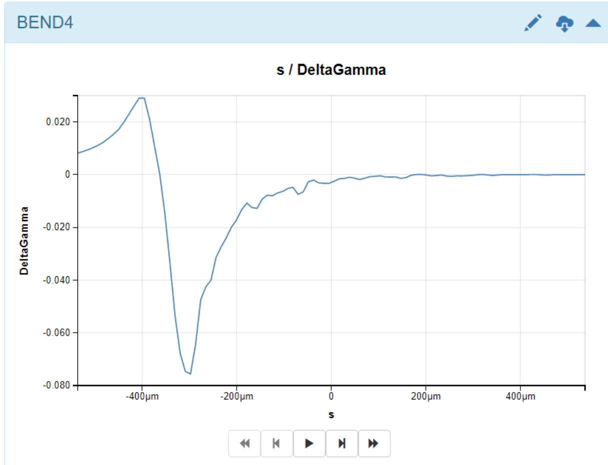

Figure 1: DeltaGamma vs. s

- Save the same 5 images to a file, for the sake of comparison.
- Write 2 or 3 sentences describing what has changed from the previous results.

b) Energy loss from CSR

1 Change the bunch charge to 0.5 nC.

2 You can do this by modifying the `total charge` parameter of the `Charge` element in your beamline (look in the lattice tab). In the command tab set `value=0` in the `alter_elements` command with `item = CSR, name = D_FODO` to make sure there is no CSR in drifts.

3 Run the simulation.

- In the `RUN_SETUP.CENTROID` panel plot `Cdelta` vs. `s`
- Save an image of this plot.
- Cdelta is the fractional momentum deviation from the central momentum of the beamline: $\frac{P-P_0}{P_0}$
- Record the final fractional momentum deviation. Do this by clicking with your mouse near the right edge of the plot window. You will see a small circle appear along the line. Get the circle close to the right vertical axis. You will see (x, y) values appear in the upper-right region of the plot window. The y-value represents Cdelta and this is what you want to record. For 0.5 nC, you should see a number close to $(-1.354 \cdot 10^{-3})$
- Write 2 or 3 sentences to describe what is happening in the plot

4 Now run the simulation for two additional bunch charges: 1 nC and 3 nC

- For each case, save an image of the plot.
- Also for each case, record the final, fractional momentum deviation (just like above).
- Based on these 3 values, consider how the average momentum loss scales with bunch charge.
- Use 2 or 3 sentences to explain your estimated scaling.

Hint: for the 3 nC case, the phase space should look like Figure 2. (You don't have to save the phase space plots. This is just a hint.)

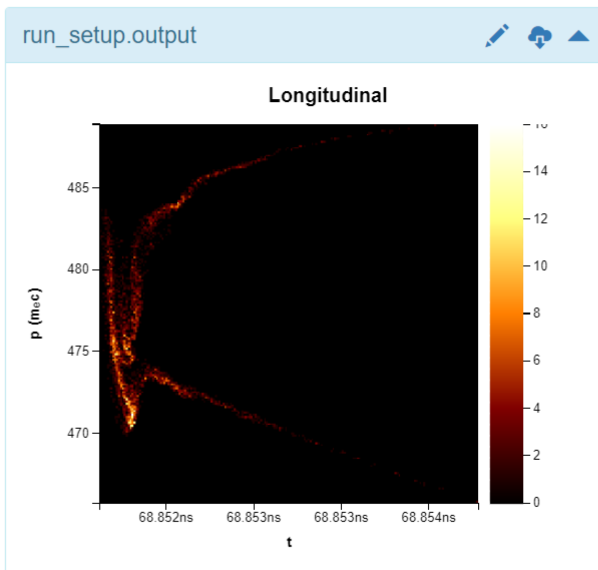

Figure 2: p vs. t