PROBLEM SET 6

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PROBLEM (1) 35 Points

Let the equation of motion for a test particle oscillating in a mismatched beam be modeled by:

$$x'' + k_\beta^2 x = \frac{-Q}{r_{b0}^6} x^5 + \frac{2\epsilon Q}{r_{b0}^2} x \cos k_B s$$

Let $\frac{x}{r_{b0}} = A \sin\psi$ & $\frac{x'}{r_{b0}} = k_\beta A \cos\psi$

where $\psi \equiv k_\beta s + \alpha$

a) CALCULATE A' & α' IN TERMS OF $A, \psi,$ & $k_B s$ (and parameters).

Let $\Psi_r' = 2k_\beta - k_B + 2\alpha_r'$

b). AVERAGE OVER RAPID VARIATIONS TO OBTAIN EQUATIONS FOR THE AMPLITUDE AND PHASE OF THE NEAR RESONANT PARTICLES. (i.e. find $A_r'(\Psi_r, A_r)$ & $\Psi_r'(\Psi_r, A_r)$).

c). DEFINE $w = A_r^2$. FIND $w'(\Psi_r, w)$ & $\Psi_r'(\Psi_r, w)$.

d). FIND THE HAMILTONIAN H, such that

$$w' = \frac{\partial H(w,\Psi_r)}{\partial \Psi_r} \quad \text{and} \quad \Psi_r' = -\frac{\partial H(w,\Psi_r)}{\partial w}$$

e). Verify that $H(w,\Psi_r)$ is a constant (i.e. $\frac{dH}{ds} = 0$).

HINT FOR PART (b): $\sin^6 x = \frac{1}{32}(10 - 15\cos 2x + 6\cos 4x - \cos 6x)$

$\cos x \sin^5 x = \frac{1}{32}(5\sin 2x - 4\sin 4x + \sin 6x)$

PROBLEM 2 10 Points

1. Suppose the space charge electric field exterior to the beam varies as:

$$E_r \sim \frac{\lambda}{4\pi\epsilon_0} \frac{r_{b0}^2}{r^3} \quad \text{instead of} \quad E_r \sim \frac{\lambda}{4\pi\epsilon_0 r}.$$

Would you expect the radial extent of a beam halo due to mismatch to be larger or smaller than the one obeying the $1/r$ dependence?

HINT: CONSIDER THE LOCATION OF THE RADIUS WHERE THE PARTICLE - ENVELOPE RESONANCE IS STRONGEST. (ASSUME HALO EXTENT IS PROPORTIONAL TO THIS RADIUS).

Problem 3, 15 Points

TCE Problem 1

S.M. Lund

1/ Envelope Radii

A/ Envelope Radius of a Nonuniform Density Beam

For a uniform density elliptical beam with envelope radii r_x and r_y

$$n(x,y) = \begin{cases} \dfrac{\lambda}{q\pi r_x r_y} & ; \quad \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} < 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

λ = line-charge density = const.

and we showed in previous problems that

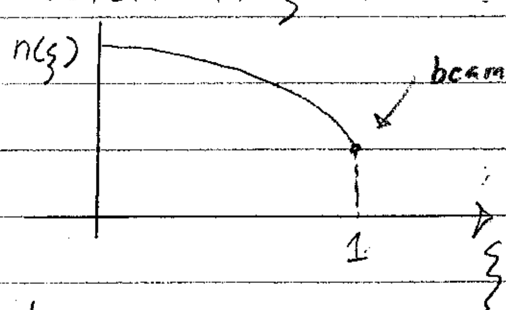
$$r_x = 2\langle x^2\rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2\rangle_\perp^{1/2}$$

where $$\langle x^2\rangle_\perp = \frac{\int d^2x \; x^2 n(x,y)}{\int d^2x \; n(x,y)}$$

If the density profile is elliptical with

$$n(x,y) = n(\xi) \quad ; \quad \xi^2 \equiv \frac{x^2}{r_{xe}^2} + \frac{y^2}{r_{ye}^2} \qquad \begin{array}{l} r_{xe} = \text{const} \\ r_{ye} = \text{const} \end{array}$$

such that $n(\xi)$ is monotonic decreasing in ξ with a sharp cutoff at $\xi = 1$



Show that

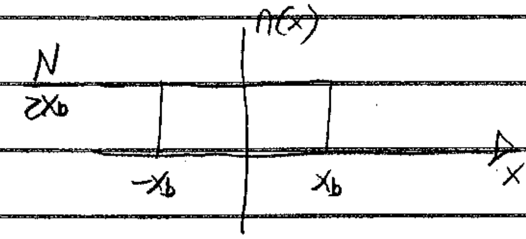
$$r_{xe} > r_x \equiv 2\langle x^2\rangle_\perp^{1/2}$$
$$r_{ye} > r_y \equiv 2\langle y^2\rangle_\perp^{1/2}$$

where $\langle x^2\rangle_\perp$ and $\langle y^2\rangle_\perp$ are defined from the nonuniform density beam. You may use steps/transforms from previous problems.

B/ "Edge" Radius Factor in a 1D Beam

For a 1D sheet beam with uniform density,

$$n(x) = \begin{cases} \frac{N}{2x_b}, & -x_b \le x \le x_b \\ 0, & \text{otherwise} \end{cases} \qquad N = \text{const}$$



x_b = edge radius

Find an edge radius coefficient F such that

$$x_b = F\langle x^2\rangle^{1/2}$$

where in 1D phase-space

$$\langle \cdots \rangle = \frac{\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \, \cdots \, f}{\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \, f} \qquad f = f(x', x; s) \text{ 1D distribution}$$

Compare F to the corresponding results for a centered 2D elliptical beam

$$r_x = 2\langle x^2\rangle^{1/2}$$
$$r_y = 2\langle y^2\rangle^{1/2}.$$

Should you expect $F = 2$? Why?

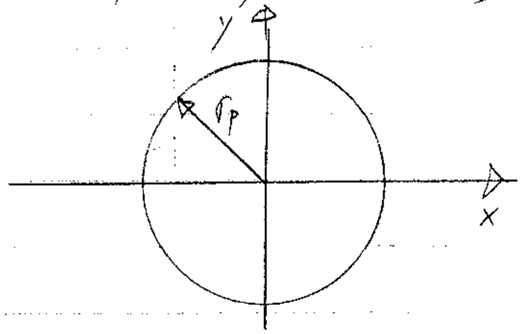
Problem 4, 20 Points

TCE Problem 2

S.M. Lund P2/

2/ Image Charges on a Cylindrical Pipe.

Consider a perfectly conducting pipe of radius r_p :



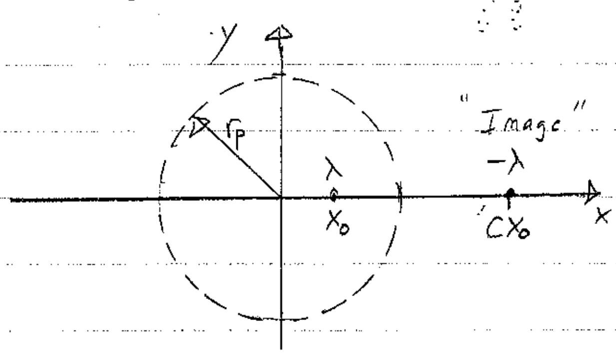
$$\frac{\partial}{\partial z} = 0$$

$$\phi(r = r_p) = \text{const}$$

A/ Show that the formula for a line-charge λ at the origin in free-space is:

$$\phi(r) = \frac{-\lambda}{2\pi\varepsilon_0} \ln r + \text{const}$$

B/ Use the formula in part A/ to show that a solution to the interior problem $|\vec{x}_\perp| < r_p$ can be found for a line charge λ at coordinate $x = x_0$ by superimposing the direct charge and an image charge at $x = Cx_0$. Calculate C for cylindrical geometry.



Images can be superimposed to obtain the Greens' Function for the 2D calculation of ϕ within the cylinder.

TCE Problem 8

S.M. Lund PI

Free Expansion of a Beam Envelope

In the absence of applied focusing forces:

$$r_x'' - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

Initial conditions $(s = s_i)$

$r_x(s=s_i) = r_{xi}$

$r_y(s=s_i) = r_{yi}$

$r_x'(s=s_i) = r_{xi}'$

$r_y'(s=s_i) = r_{yi}'$

Part I: $Q = 0$, $\varepsilon_x \neq 0$, free expansion without space charge.

a) Show that the envelope Hamiltonian satisfies:

$$\frac{r_x'^2}{2} + \frac{\varepsilon_x^2}{2 r_x^2} = \text{const.}$$

r_y equation analysis same.

b) Show that the equation in a) can be written as an integral from the initial condition to solve for the free expansion as a function of s. You do not need to carry out the integration explicitly.

Part II $\varepsilon_x = 0$, $Q \neq 0$ free expansion without emittance

c) Show using $r_\pm \equiv (r_x \pm r_y)/2$ that the coupled envelope equations reduce to:

$$r_+'' - \frac{Q}{r_+} = 0 , \qquad r_-'' = 0$$

and satisfy

$$\frac{r_+'^2}{2} - Q \ln r_+ = \text{const} , \qquad r_- = C_1 s + C_2$$

$C_{1,2}$ constants

TCE Problem 8

d) Argue that for finite $Q \neq 0$ that the free expansion without emittance will be more rapid than the free expansion without space charge ($Q = 0$, $\varepsilon_x, \varepsilon_y \neq 0$) when the beam expands sufficiently.