Problem Set #1 BARNARD & LUND USPAS JUNE 2011

20 POINTS

1. CONSIDER A ROUND UNIFORM ION BEAM WITH A CURRENT OF 1 AMPERE OF K^+ ($A = 39$) IONS, A KINETIC ENERGY OF 2 MeV, A BEAM RADIUS OF 2 cm AND NORMALIZED EMITTANCE OF 1 mm-mrad.

CALCULATE, FOR THESE BEAM PARAMETERS: (to 1 or 2 significant figures)

a) $\beta = v_0/c$

b) n = number density of ions in beam

c) $kT_\perp$ = transverse temperature (express in eV)

d) λ_D = transverse Debye length

e) Q = generalized perveance

f) Λ = plasma parameter

g) $\Delta\Phi$ = potential difference between center and edge of beam.

FOR REFERENCE

$c = 3 \times 10^8$ m/s [speed of light]

$e = 1.6 \times 10^{-19}$ C [electron charge]

$k_B = 1.38 \times 10^{-23}$ J K^{-1} [Boltzmann's const]

$\epsilon_0 = 8.854 \times 10^{-12}$ F m^{-1} [permittivity of free space]

$m_u = 1.66 \times 10^{-27}$ kg

$m_u c^2 = 931.1 \times 10^6$ eV

[atomic mass unit]

.0
POINTS

(2) Show that

$$\left\langle r \frac{\partial \varphi}{\partial r} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0}$$

for a charge distribution in which $\rho(r,\theta) = \rho(r)$ only.

[λ = line charge density = $\int_0^\infty 2\pi r \, \rho(r) \, dr$

$\langle g \rangle = \frac{1}{\lambda} \int_0^\infty g(r) \, 2\pi r \, \rho(r) \, dr$]

where g is any beam quantity

PROBLEM (3)
[20 POINTS]

Let the single particle equation of motion of a particle be:

$$x'' = -\alpha(s)\, x^n$$

Here x is the usual transverse coordinate and s is the longitudinal coordinate.

Calculate the derivative with respect to s of the square of the emittance $\epsilon^2 = 16[\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2]$.

Express $\frac{d\epsilon^2}{ds}$ in terms of $\langle x^2\rangle$, $\langle xx'\rangle$, $\langle x'x^n\rangle$ and $\langle x^{n+1}\rangle$.

For what value of n is $\frac{d\epsilon^2}{ds}$ identically zero?

Problem 4 20 Points

S. M. Lund P1/

JPD Problem 1 – Larmor Frame

For a uniform solenoidal channel:

$$B_z^a(s) = B_0 = \text{const}$$

with no acceleration

$$\gamma_b \beta_b = \text{const}$$

and an axisymmetric ($\partial/\partial\theta = 0$) beam with

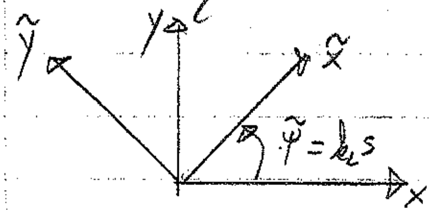
$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\vec{x}_\perp}{r} \qquad r = \sqrt{x^2+y^2}$$

the particle equations of motion reduce to:

$$x'' = \frac{qB_0}{m\gamma_b\beta_b c} y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{x}{r}$$

$$y'' = -\frac{qB_0}{m\gamma_b\beta_b c} x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{y}{r}$$

a) Parallel steps taken in the class notes to transform the equations of motion to a co-rotating frame:



k_L = const = Larmor wavenumber

$$\tilde{x} = x\cos(k_L s) + y\sin(k_L s)$$

$$\tilde{y} = -x\sin(k_L s) + y\cos(k_L s)$$

Find an expression for k_L to reduce the equations of motion to the decoupled form:

TPD Problem 1

S. M. Lund

$$\tilde{x}'' + \kappa \tilde{x} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \kappa \tilde{y} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

and identify $\kappa = \text{const.}$

Hint:
The transformation can be carried out directly. But you may find the algebra simpler using complex coordinates as in the class notes:

$$\underline{z} = x + iy \qquad i = \sqrt{-1} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$\tilde{\underline{z}} = \tilde{x} + i\tilde{y}$$

b) If the direction of the magnetic field is reversed:
$$B_0 \rightarrow -B_0$$
how will the dynamics be influenced?

c) Neglect space-charge:
$$\phi = 0$$
and sketch a typical orbit in the rotating Larmor frame. Will this orbit appear more complicated in the Laboratory frame? Why?

Bonus: Sketch the orbit taking advantage of simple choices of initial conditions that can always be made through choice of coordinates.

Problem 5 10 Points

TPD Problem 2 S.M. Lund P2/

2/ Derive the 2×2 transfer matrices $M(s|s_i)$ advancing the particle orbit $x(s)$, $x'(s)$ from the initial conditions at $s = s_i$

$$x(s_i) \equiv x_i$$
$$x'(s_i) \equiv x_i'$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s|s_i) \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

for the following:

a) Free Drift

equation: $x'' = 0$

b) Continuous focusing:

equation: $x'' + k_{\beta 0}^2 x = 0$ $\qquad k_{\beta 0}^2 = \text{const.} > 0.$

c) Quadrupole focusing:

equation: $x'' + \kappa_q x = 0$ $\qquad \kappa_q = \text{const} > 0$

d) Quadrupole defocusing:

equation: $x'' - \kappa_q x = 0$ $\qquad \kappa_q = \text{const} > 0$

e) Verify in cases a) – d) that the Wronskian W satisfies:

$$W = \det M(s|s_i) = 1$$

TPD Problem 4   Problem 6

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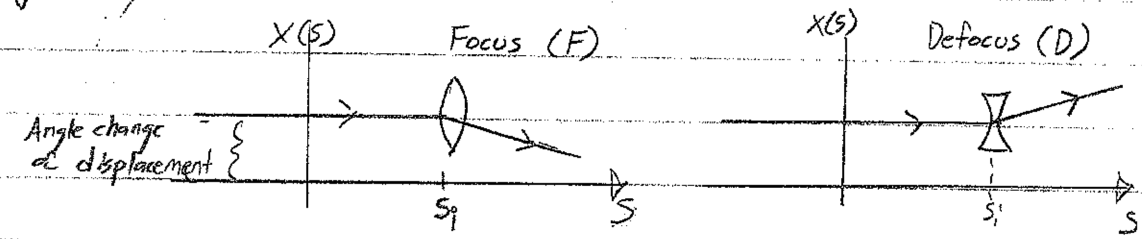
4/ A thin lense changes the angle of a particle trajectory but not the coordinate:



This action can be specified by transfer matrices applied at $s = s_i$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s|s_i) \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

Focusing:

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$f > 0$

Defocusing:

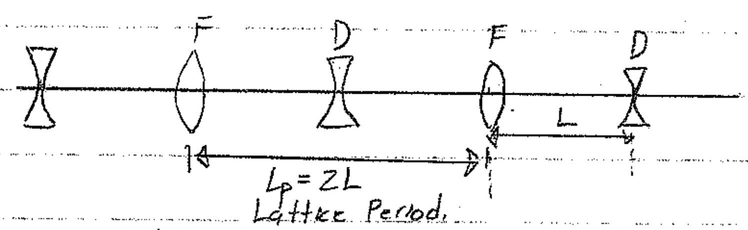
$$M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$f > 0$

From TPE Problem 3 a free-space drift of length L has a transport Matrix:

$$M_O = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Consider a lattice of period $2L$ made up of equally spaced F and D lenses with equal values of f.



This is the simplist "FODO" alternating gradient lattice!

a) Use the transfer matrix analysis developed in class to find the range of f for which the particle orbit is stable.

b) Calculate $\cos\sigma_0$ where σ_0 is the particle phase advance.

c) For the case of of f chosen to correspond to the stability limit, sketch the motion of a particle with initial condition

$$\lim_{s\to s_i^-} x(s) = x_0$$
$$\lim_{s\to s_i^-} x'(s) = x_0/L$$

where $s = s_i$ is the axial location of a focusing thin lens kick, and $s \to s_i^-$ is just before the kick. Sketch the particle orbit for focusing strength slightly larger than the stability limit. Superimpose the orbit sketch on a diagram of the lattice (see below):