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S4 Collective Modes on a KV Equilibrium Beam

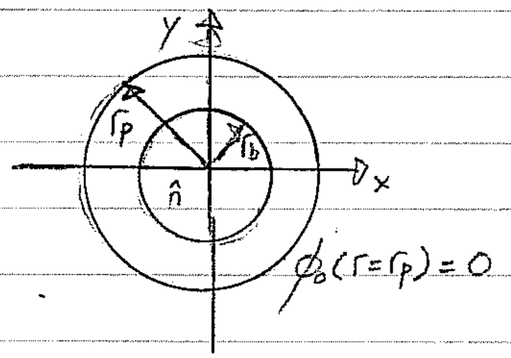
Here we take a KV equilibrium distribution with

$$\boxed{f_0(H_0) = \frac{\hat{n}}{2\pi}\,\delta\left[H_0 - \frac{\varepsilon_x^2}{2r_b^2}\right]}$$

$\hat{n}$ = constant density of KV equilibrium
ε_x^2 = x-emittance.
r_b = equilibrium beam radius.

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

$$H_0 = \frac{1}{2}\vec{x}_\perp'^2 + \frac{k_{\beta 0}^2}{2}\vec{x}_\perp^2 + \frac{q\,\phi_0}{m\gamma_b^3\beta_b^2c^2}$$



and assume small-amplitude axisymmetric ($\partial/\partial\theta=0$) perturbations with normal mode form:

$$\boxed{\begin{aligned}\delta f(\vec{x}_\perp,\vec{x}_\perp',s) &= \delta f(r,\vec{x}_\perp',k)\,e^{-iks}\\ \delta\phi(\vec{x}_\perp,s) &= \delta\phi(r,k)\,e^{-iks}\end{aligned}}$$

k = const (mode eigenfrequency)

The equilibrium characteristics in the core of the KV beam can be expressed as:

$$\boxed{\begin{aligned} r^2(\tilde{s}) &= r^2\cos^2[k_\beta(\tilde{s}-s)] + \frac{r r'}{k_\beta}\cos\psi \sin[2k_\beta(\tilde{s}-s)] \\ &\quad + \frac{r'^2}{k_\beta^2}\sin^2[k_\beta(\tilde{s}-s)] \\ x(\tilde{s}=s) &= r\cos\theta \quad ; \quad x'(\tilde{s}=s) = r'\cos\theta_p \\ y(\tilde{s}=s) &= r\sin\theta \quad ; \quad y'(\tilde{s}=s) = r'\sin\theta_p \\ \psi &\equiv \theta - \theta_p \\ k_\beta &= \left(k_{\beta 0}^2 - \frac{Q}{r_b^2}\right)^{1/2} = \frac{\varepsilon_x}{r_b^2} \end{aligned}}$$

Depressed β-tron wavenumber of particle oscillations

These results can be inserted into the characteristic equation

$$\delta f(\vec{x}_\perp, \vec{x}_\perp', s) = \frac{q}{m\gamma_b^3\beta_b^2c^2}\int_{-\infty}^{s} d\tilde{s}\; \frac{\partial \delta\phi(\vec{x}_\perp(\tilde{s}))}{\partial \vec{x}_\perp(\tilde{s})} \cdot \frac{\partial}{\partial \vec{x}_\perp'(\tilde{s})} f_0\left(H_0(\vec{x}_\perp(\tilde{s}), \vec{x}_\perp'(\tilde{s}))\right)$$

to derive an expression for $\delta f(r, \vec{x}_\perp')$. This expression can then be inserted into the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi(r)}{\partial r}\right) = -\frac{q}{\varepsilon_0}\int d^2x'\, \delta f(r, \vec{x}_\perp')$$

to derive a linear eigenvalue equation for $\delta\phi(r)$:

A significant amount of manipulation obtains the following form for the eigenvalue equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\delta\phi(r) = \frac{\hat{\omega}_p^2}{\gamma_b \beta_b^2 c^2}\,\Theta(r_b - r)\,\frac{1}{r_\perp'}\frac{\partial}{\partial r_\perp'} I_{orb}(r, r_\perp', k)\Bigg|_{r_\perp'^2 = \frac{\varepsilon_x^2}{r_b^2}\left(1-\frac{r^2}{r_b^2}\right)} \quad ①$$

$$+ \frac{\hat{\omega}_p^2/(\gamma_b \beta_b^2 c^2)}{\varepsilon_x^2/r_b^2}\,\delta(r - r_b)\left[\delta\phi + I_{orb}(r, r_\perp', k)\right]_{r_\perp' = 0} \quad ②$$

Subject to: $\delta\phi(r = r_b) = 0$, $\hat{\omega}_p^2 = \frac{q^2 \hat{n}}{\varepsilon_0 m}$ = Plasma Freq. Squared.

where:

$$\Theta(r_b - r) \equiv \begin{cases} 1 & r_b > r \\ 0 & r_b < r \end{cases} \quad \text{Heaviside step function}$$

$$I_{orb}(r, r_\perp', k) = ik\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}\int_{-\infty}^{s} d\tilde{s}\;\delta\phi(r(\tilde{s}), k)\, e^{-ik(\tilde{s}-s)}$$

Orbit integral.

Note:

- Term ① $\propto \Theta(r_b - r)$ is a <u>body-wave perturbation</u> existing only in the core ($r < r_b$) of the equilibrium beam.
- Term ② $\propto \delta(r - r_b)$ is a <u>surface-wave perturbation</u> existing only at the edge ($r = r_b$) of the equilibrium beam.
- The orbit integral $I_{orb}(r, r_\perp', k)$ depends on both $\delta\phi$ and the <u>eigenfrequency</u> k.

The Poisson equation has become a <u>linear</u> integro-differential eigenvalue equation fixing the mode perturbed potential $\delta\phi$ and eigenfrequency k.

Gluckstern Mode Solution

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This eigenvalue equation is difficult, but it has been solved analytically

- A finite polynomial in r^2 expansion of $\delta\phi$ for $r<r_b$ can satisfy the equation (terms truncate)
- Expansions are inserted into the characteristic integrals and coefficients are identified power-by-power in r^2, and assembled.

Solution (after much analysis)

Eigenfunction:

$$\boxed{\delta\phi_n(r) = \begin{cases} \dfrac{A_n}{2}\left[P_{n-1}(1-2r^2/r_b^2) + P_n(1-2r^2/r_b^2)\right]; & 0\le r<r_b \\ 0 & ; r_b<r\le r_p \end{cases}}$$

$n = 1, 2, 3, \ldots$ radial mode index

$A_n = \text{const.}$ linear mode amplitude.

$P_n(x)$ n'th order Legendre Polynomial

Dispersion Relation:

Each n-labeled eigenfunction has $2n$ (degenerate) "eigenfrequencies" k satisfying an nth degree polynomial in k^2 dispersion relation.

$$\boxed{2n + \frac{1-(\sigma/\sigma_0)^2}{(\sigma/\sigma_0)^2}\left[B_{n-1}\left(\frac{k/k_{\beta0}}{\sigma/\sigma_0}\right) - B_n\left(\frac{k/k_{\beta0}}{\sigma/\sigma_0}\right)\right] = 0}$$

where:
$$\frac{\sigma}{\sigma_0} = \frac{k_\beta}{k_{\beta0}} = \frac{(k_{\beta0}^2 - Q/r_b^2)^{1/2}}{k_{\beta0}}$$
and

$$\boxed{B_n(\alpha) \equiv \begin{cases} 1 & n=0 \\ \dfrac{[(\alpha/2)^2-0^2]}{[(\alpha/2)^2-1^2]}\cdot\dfrac{[(\alpha/2)^2-2^2]}{[(\alpha/2)^2-3^2]}\cdots\cdots\dfrac{[(\alpha/2)^2-(n-1)^2]}{[(\alpha/2)^2-n^2]} & n=1,3,5,\ldots \\ \dfrac{[(\alpha/2)^2-1^2]}{[(\alpha/2)^2-2^2]}\cdot\dfrac{[(\alpha/2)^2-3^2]}{[(\alpha/2)^2-4^2]}\cdots\cdots\dfrac{[(\alpha/2)^2-(n-1)^2]}{[(\alpha/2)^2-n^2]} & n=2,4,6,\ldots \end{cases}}$$

Properties:

Radial Eigenfunction:

- Vanishes outside the equilibrium beam edge ($r > r_b$).
- Has $n-1$ nodes with $\delta\phi = 0$ within the equilibrium beam ($r < r_b$).
- Each n labeled eigenfunction has $2n$ distinct frequencies

Corresponding perturbed density can be calculated from Poisson's equation:

$$\boxed{\begin{aligned} \delta n_n &= \delta n_n(r) e^{-iks} \\ \delta n_n(r) &= -\frac{\varepsilon_0}{q}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \delta\phi_n}{\partial r}\right) \end{aligned}}$$

- Find that the perturbed density of the mode is more larger near the outer ($r \lesssim r_b$) edge of the beam for larger n.

Eigenfunction Form

Mode number n	$\delta\phi_n / A_n$ (potential)	δn_n (density, scaled units)
1	$1-\tilde{r}^2$	1
2	$1-4\tilde{r}^2+3\tilde{r}^4$	$4(1-3\tilde{r}^2)$
3	$1-9\tilde{r}^2+18\tilde{r}^4-10\tilde{r}^6$	$9(1-8\tilde{r}^2+10\tilde{r}^4)$
4	$1-16\tilde{r}^2+60\tilde{r}^4-80\tilde{r}^6+35\tilde{r}^8$	$16(1-15\tilde{r}^2+45\tilde{r}^4-35\tilde{r}^6)$
⋮	⋮	⋮

$$\tilde{r} \equiv r/r_b$$

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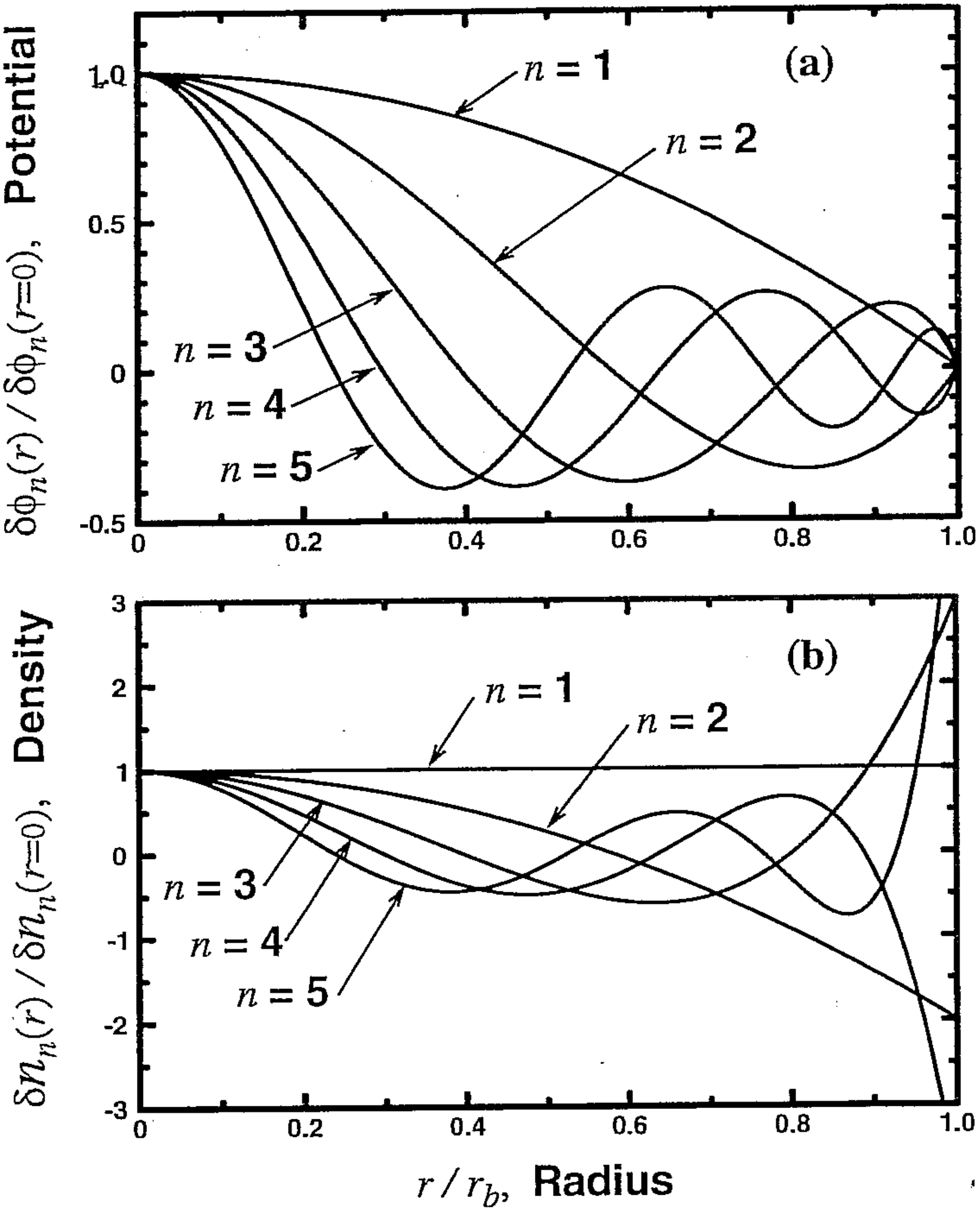
Radial Eigenfunction



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Perturbations should introduce no net charge into the system:

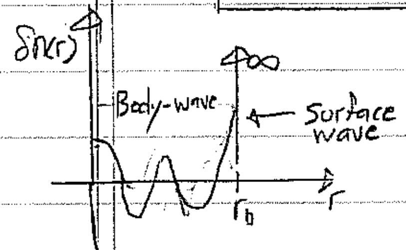
$$2\pi \int_0^{r_p} dr\, r\, \delta n(r) = 0$$

The $r < r_b$ component of the perturbations are not the only terms present. For the $r < r_b$ eigenfunctions calculated: $\int_0^{r_d} dr\, r\, \delta n_n(r) \neq 0$. A more detailed analysis shows that:

$$\boxed{\delta n_n(r) = \delta n_n(r)\Big|_{body} \Theta(r_b - r) + \delta n_n\Big|_{surface} \frac{r_b^2}{r} \delta(r - r_b)}$$

body-wave perturbation in core of equilibrium

singular charge conserving beam edge adjustment at $r = r_b$



where:

$$\delta n_n\Big|_{body} = -\frac{\varepsilon_0}{q} \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \delta \phi_n}{\partial r} \right)$$

$$\delta n_n\Big|_{surface} = \text{const} \times (-1)^n n A_n$$

To linear order this is equivalent to:

$$\boxed{n(r) = [\hat{n} + \delta n_n(r)\big|_{body}]\, \Theta[r_b + \delta r_b - r]}$$

$$\delta r_b = \text{const} \times (-1)^n n A_n$$ readjustment of beam edge radius.

Dispersion Relation

- Polynomial in k^2 $\Rightarrow$ $\pm k$ solutions and therefore there will be unstable growing perturbations if k is complex:

$$\delta\phi \sim \delta\phi_n(r)\, e^{-iks}$$

$$k = k_r \pm i k_I \qquad \begin{aligned} k_r &= \text{real part} \\ k_I &= \text{imaginary part} \end{aligned}$$

For the unstable branch:

$$\delta\phi \sim \delta\phi_n(r)\, e^{-ik_r s} \cdot e^{|k_I| s} \quad \Rightarrow \text{exponential growth.}$$

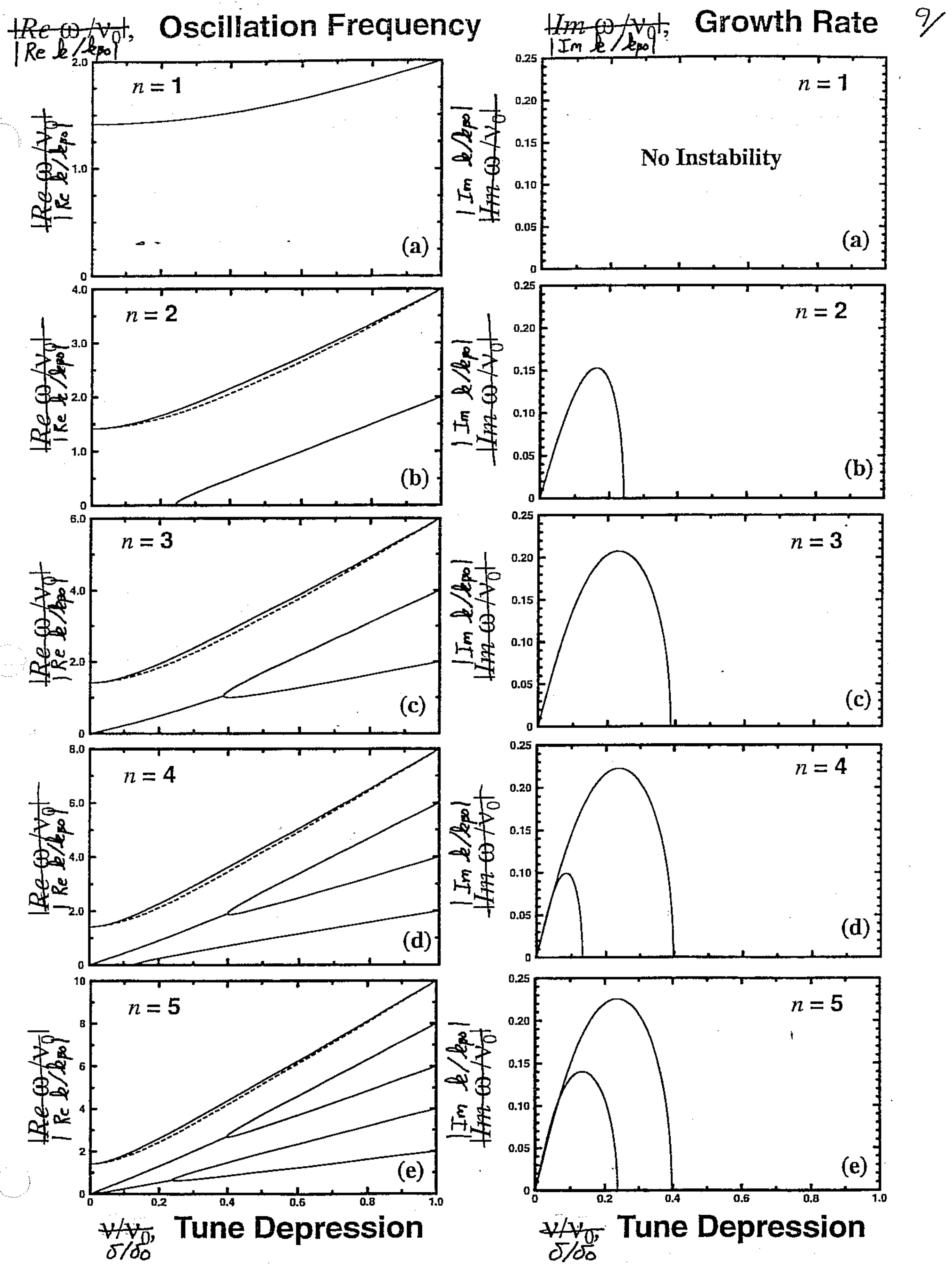
- $|k|$ is a function of n and σ/σ_0 <u>only</u>.
  - $0 \leq \sigma/\sigma_0 \leq 1$

0: strongest possible space charge. $\qquad 1$: zero space-charge.

  - <u>Instabilities will occur over a range of σ/σ_0 and will turn off for σ/σ_0 large enough</u> (weak space-charge). KV beam is always stable for zero space-charge since orbits are stable.

<u>Dispersion Relations:</u>

Mode number n	Dispersion relation
1	$(k/k_{\beta 0})^2 - 2(1 + \sigma^2/\sigma_0^2) = 0$
2	$(k/k_{\beta 0})^4 - 2(1 + 9\sigma^2/\sigma_0^2)(k/k_{\beta 0})^2 - 4(\sigma^2/\sigma_0^2)(1 - 17\sigma^2/\sigma_0^2) = 0$
$\vdots$	$\vdots$ Rapidly more complicated!



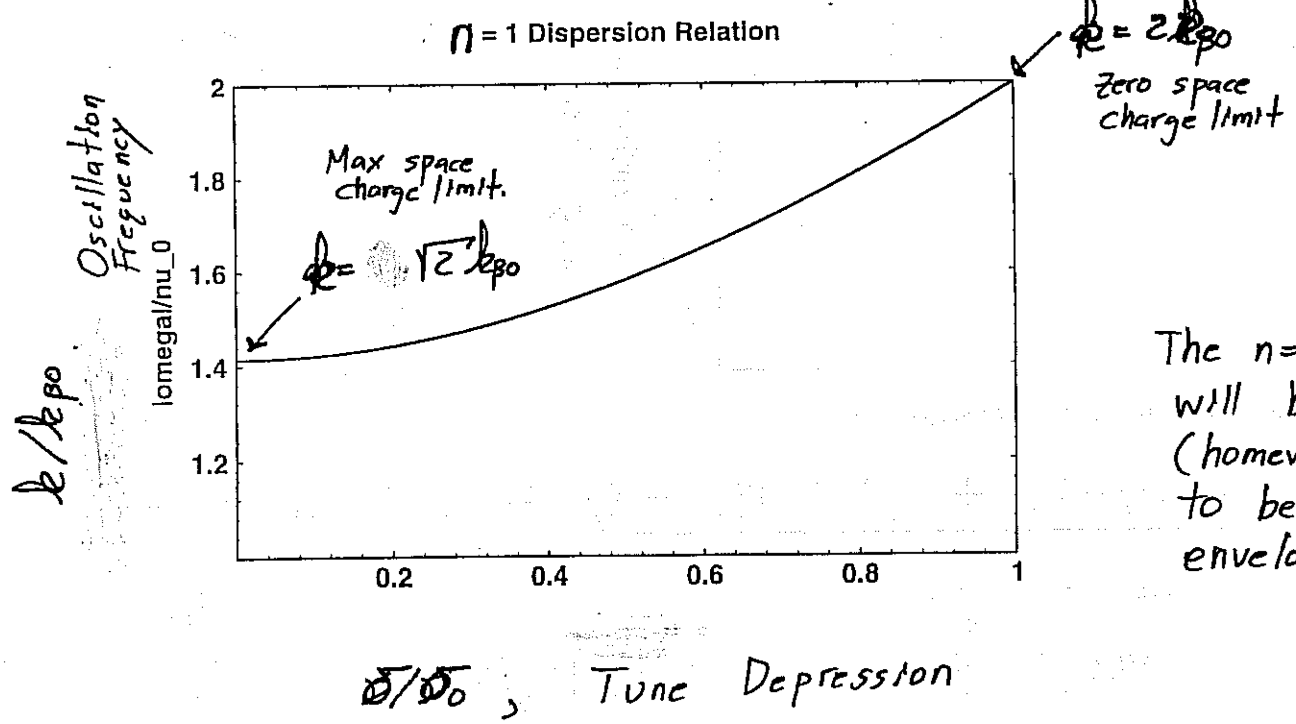
~~$|Re\ \omega/\nu_0|$~~, $|Re\ k/k_{\beta 0}|$ Oscillation Frequency
~~$|Im\ \omega/\nu_0|$~~, $|Im\ k/k_{\beta 0}|$ Growth Rate
n = 1
n = 2
n = 3
n = 4
n = 5
No Instability
(a)
(b)
(c)
(d)
(e)
~~ν/ν_0~~, δ/δ_0 Tune Depression
~~ν/ν_0~~, δ/δ_0 Tune Depression

Fig. 8

Example: $n = 1$, Envelope Mode

$$\delta\phi_1 = \begin{cases} A_1[1-(r/r_b)^2], & 0 \le r \le r_b, \\ 0, & r_b \le r \le r_p, \end{cases}$$

$$(k/k_{\beta 0})^2 = 2 + 2(\bar{\sigma}/\bar{\sigma}_0)^2$$



The $n=1$ mode will be shown (homework) to be the usual envelope mode

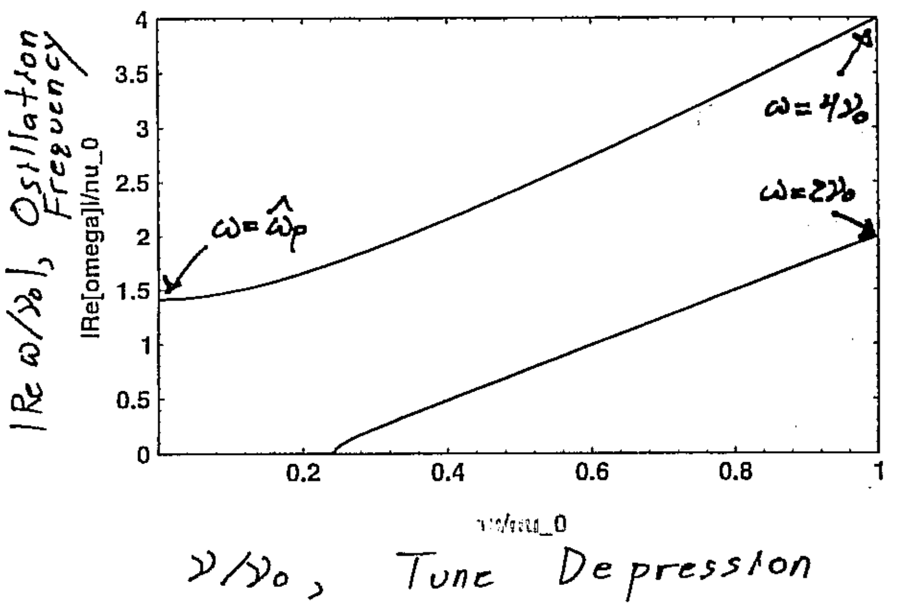
Oscillation Frequency

Kinetic Theory – Transverse Gluckstern Modes (7)

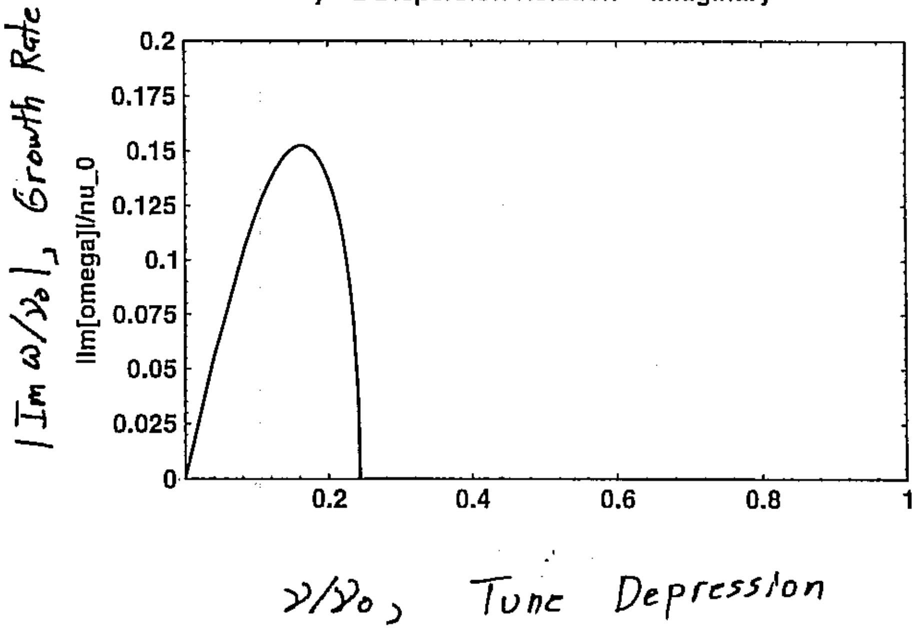
Example: $n = 2$ Mode

$$\delta\phi_2 = \begin{cases} A_2[1 - 4(r/r_b)^2 + 3(r/r_b)^4], & 0 \le r \le r_b, \\ 0, & r_b \le r \le r_p, \end{cases}$$

$$(\omega/\nu_0)^2 = 1 + 9(\nu/\nu_0)^2 \pm \sqrt{1 + 22(\nu/\nu_0)^2 + 13(\nu/\nu_0)^4}$$



Oscillation Frequency



Growth Rate

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Discussion:

As might be expected on physical grounds, the singular KV distribution drives numerous, strong, collective instabilities. This implies that the KV model is suspect since real beams are often transported where the KV model would predict strong instability. However:

- Low-order KV features (envelope modes) are correct and well verified.
- Higher order collective modes observed on intense beam cores often look similar to the KV model predictions in density/potential etc, but are not unstable.

How is this situation resolved? A partial answer was suggested by a fluid model developed by Lund and Davidson. In this model:

- Density and tempeture profiles (i.e., low order features) of the KV model were preserved.
- The singular phase-space structures were eliminated.

A stability analysis obtained:

Mode Eigenfunction: $(r<r_b)$

$$\delta\phi_n = \frac{A_n}{2}\left[P_{n-1}\left(1 - 2\frac{r^2}{r_b^2}\right) + P_n\left(1 - 2\frac{r^2}{r_b^2}\right)\right]$$

Mode Dispersion Relation:

$$\left(\frac{k}{k_{\beta 0}}\right)^2 = 2 + 2\left(\frac{\sigma}{\sigma_0}\right)^2 (2n^2 - 1)$$

$$n = 1, 2, 3, \ldots\ldots$$

Features of fluid model:

- Identical radial eigenfunction to the full kinetic theory
- Fluid mode dispersion relation predicts stability for all modes and closely tracks the (stable) high frequency branch of the KV dispersion relation for the full range of space charge strength $0 \leq \sigma/\sigma_0 \leq 1$
  - Fluid mode dispersion relation plotted dashed on KV mode plots.
  - The $n=1$ fluid envelope mode is identical to the KV envelope mode.

Since the fluid model reproduces the coarse macroscopic features of the KV model – which can be a good approximation at high space-charge intensities, this implies:

- KV model mode eigenfunctions should roughly model those of intense beams with smooth distributions.
- Oscillation frequencies may be close to the (stable) high frequency KV mode branch
  - May be other lower frequency branches that are also physical.
- Many high-order KV instabilities may be of little relevance to real beams.
  - Low order (envelope and maybe others) can be relevant.

The real issue for high intensity collective modes may not be higher order KV instabilities but if low-order collective modes can:

- Be driven unstable by periodic (s-varying) focusing structures in machine lattices, errors in rings, etc.
- Drive the production of beam halo, etc.

References:

Material on the kinetic stability of KV beams is found mostly in journals.

Original references

Gluckstern, Proc. 1970 Proton Linac Conference, Nat. Accel. Lab., pg. 811 – First KV mode analysis.

T.F. Wang and L. Smith, Part. Accel. 12, 247 (1982). – Simplified (closed form) mode eigenfunction and dispersion relation.

Interpretation of Branches, Mode Structure, KV Fluid Stability

S.M. Lund and R.C. Davidson, Physics of Plasmas 5, 3028 (1998). Detailed analysis of eigenfunctions, dispersion relations, etc. in appendices. Fluid mode analysis and interpretations of KV modes.

Other papers by Hoffmann, Gluckstern, and others. Hoffmann et. al. analyzed KV in periodic focusing lattices.