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USPAS
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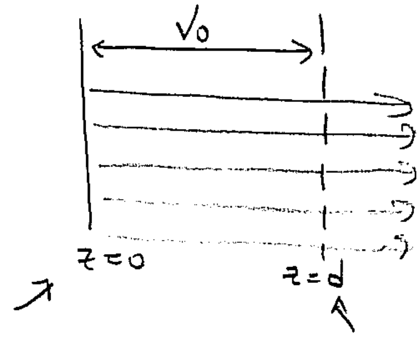
Injectors and longitudinal physics -- III

1. Longitudinal cooling from acceleration
2. Longitudinal instability
3. Bunch compression
4. Neuffer distribution

LONGITUDINAL COOLING

1. DURING INJECTION BEAM UNDERGOES LARGE LONGITUDINAL EXPANSION
2. $T_{\perp 0} = T_{\parallel 0}$ AT SOURCE, BUT $T_{\perp} \neq T_{\parallel}$ AFTER ACCELERATION
3. IMPLICATIONS FOR BEAM STABILITY AND EMITTANCE EVOLUTION

CONSIDER 1 D DIODE:



AT SOURCE

$$E_0 = \frac{p_{z0}^2}{2m}$$

$$\Delta E_{\parallel 0} \equiv \frac{\langle p_{z0}^2 \rangle}{2m} = \frac{1}{2} kT_{\parallel 0}$$

AT END OF DIODE

$$E_f = qV_0 + \frac{p_{z0}^2}{2m} = \frac{p_{zf}^2}{2m}$$

$$\Delta E_{\parallel f} = \Delta E_{\parallel 0} \neq \frac{1}{2} kT_f$$

SINCE $E_{\parallel} = \frac{p_z^2}{2m} \Rightarrow \Delta E_{\parallel} = \frac{2 p_z \Delta p_z}{2m}$

$$\frac{\Delta E}{E} = \frac{2 \Delta p_z}{p_z}$$

$$\frac{1}{2} kT_f \cong \frac{\Delta p_{zf}^2}{2m} = \left(\frac{p_{zf} \Delta E_f}{2 E_f} \right)^2 \frac{1}{2m} = \frac{\Delta E_f^2}{4 E_f} = \frac{kT_0}{2} \left[\frac{1}{2} \frac{kT_0}{qV_0} \right]$$

$$\Rightarrow \boxed{kT_f = \frac{1}{2} kT_0 \left[\frac{kT_0}{qV_0} \right]}$$

$$kT_f = \frac{1}{2} kT_0 \left[\underbrace{\frac{kT_0}{qV_0}}_{\ll 1} \right]$$

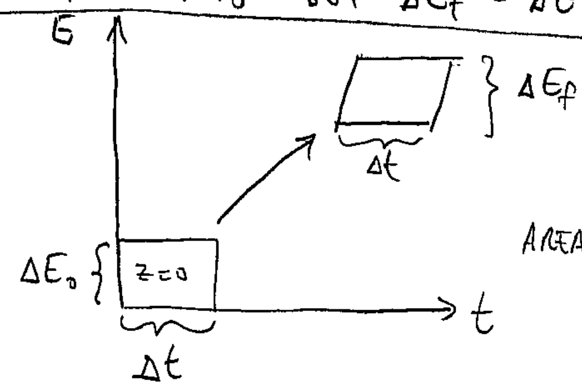
EXAMPLE $1000°C \Leftrightarrow 0.1\ eV$

FOR $V_0 = 1\ MeV$

$kT_0 = 0.1\ eV$

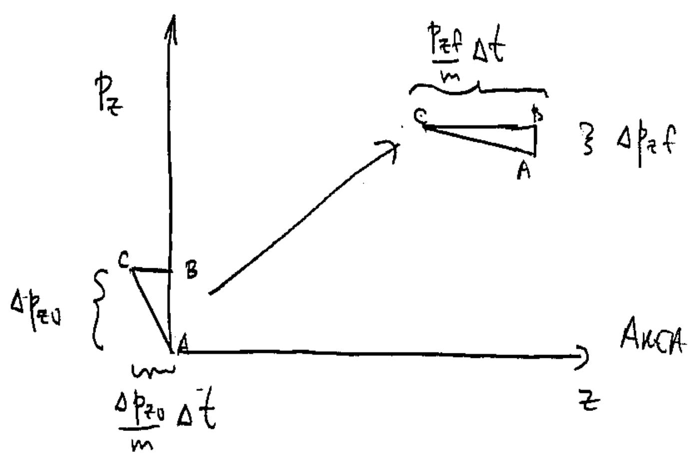
$kT_f = 5. \times 10^{-9}\ eV$

$\boxed{\text{HOW CAN } kT_f \ll kT_0 \text{ BUT } \Delta E_f = \Delta E_0 \text{?}}$



AREA IS CONSERVED

(PULSE DURATION STAYS THE SAME.)



(BUNCH LENGTH GROWS)

AREA IS CONSERVED

$$\frac{1}{2} \frac{\Delta p_{z0}^2}{m} \Delta t = \frac{1}{2} \Delta p_{zf} \left(\frac{p_{zf}}{m} \right) \Delta t$$

$$\Rightarrow \Delta p_{zf} = \frac{\Delta p_{z0}^2}{p_{zf}}$$

$$\Rightarrow kT_f = \frac{1}{2} kT_0 \left(\frac{kT_0}{qV_0} \right)$$

CHANGE IN NOTATION

NOTE: $\overline{z'} \equiv \left\langle \frac{dz}{ds} \right\rangle$; $\quad s \equiv v_0 t$

Let $\quad u = \left\langle \frac{dz}{dt} \right\rangle$; $\quad$ then $\quad u = v_0 \overline{z'}$

= fluid velocity in comoving frame

So

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}\left(\lambda \overline{z'}\right) = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z}(\lambda u) = 0}$$

$$\& \quad \frac{\partial \overline{z}'}{\partial s} + \overline{z'}\frac{\partial}{\partial z}\overline{z'} + \frac{1}{\lambda}\frac{\partial}{\partial z}\left(\lambda \Delta z'^2\right) = \overline{z''}$$

$$\Rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{1}{\lambda}\frac{\partial}{\partial z}\left(\lambda\left[\langle v_z^2 \rangle - u^2\right]\right) = \overline{\ddot{z}}$$

Since $p_z = m\int_{-\infty}^{\infty} n\left[v_z^2 - u^2\right] dv_z$ $\quad$ where $n = \frac{\lambda}{\pi r_b^2}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{\pi r_b^2}{m\lambda}\frac{\partial}{\partial z}p_z = \overline{\ddot{z}}} \quad \text{where } \ddot{z} = \frac{d^2 z}{dt^2} = \frac{F_z}{m}$$

"LONGITUDINAL" OR "RESISTIVE WALL" INSTABILITY

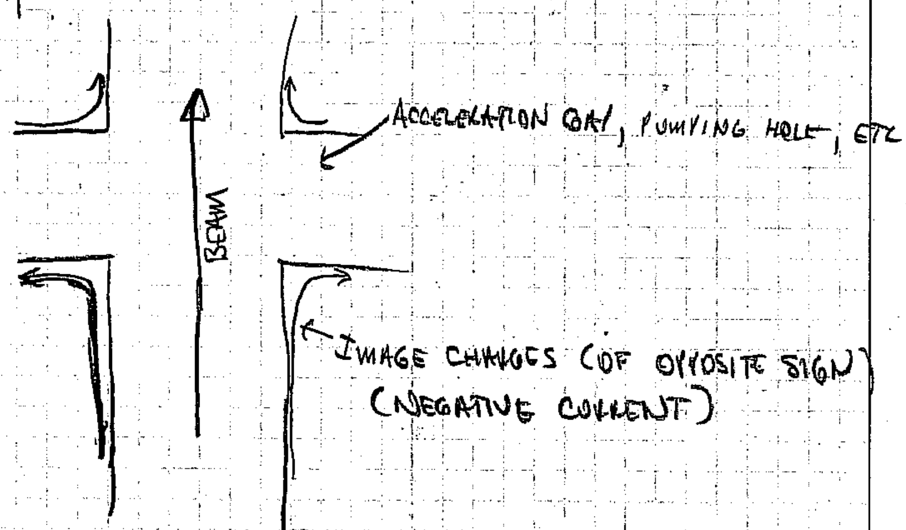
Let us return to the 1-D FLUID EQUATIONS

SEE REISER 6.3.2.
CALLAHAN-MILLER, PH.D. DISSERTATION, U.C. DAVIS, 1994

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z}\lambda u = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \underbrace{\frac{1}{\rho}\frac{\partial p_z}{\partial z}}_{\text{IGNORE AGAIN}} = \frac{-g q}{4\pi\epsilon_0 m v_0^2}\frac{\partial \lambda}{\partial z} + \underbrace{\frac{qE_z}{m}}_{\text{EXTERNALLY GENERATED}}$$

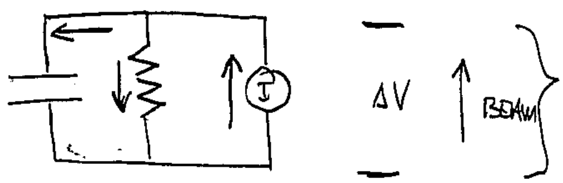
AS BEAM PASSES CONDUCTING SURFACE IMAGE CHARGE AND CURRENT INTERACTS WITH BEAM. HIGHLY GEOMETRY DEPENDENT.



CAN BE CALCULATED APPROXIMATELY USING CIRCUIT MODEL.

RESISTIVITY IN WALL, AND COMPLICATED ELECTRON FLOW PATTERNS ~~CREATE~~ A RETARDING ELECTRIC FIELD ON BEAM.

MODEL OF IMPEDANCE (IN LONG WAVELENGTH REGIME)



ONE MODULE OF MANY, EACH SEPARATED BY DISTANCE L

$$I = C\frac{d\Delta V}{dt} + \frac{\Delta V}{R} \qquad I = [CL]\frac{d\Delta V/L}{dt} + \frac{\Delta V/L}{R/L}$$

$$E = -\frac{\Delta V}{L} \qquad C^+ = CL \qquad R^* = \frac{R}{L}$$

LET $I = I_0 + I_1 e^{-i\omega t}$ $\qquad E = E_0 + E_1 e^{-i\omega t}$

$$I_1 = i\omega C^+ E_1 - \frac{E_1}{R^*}$$

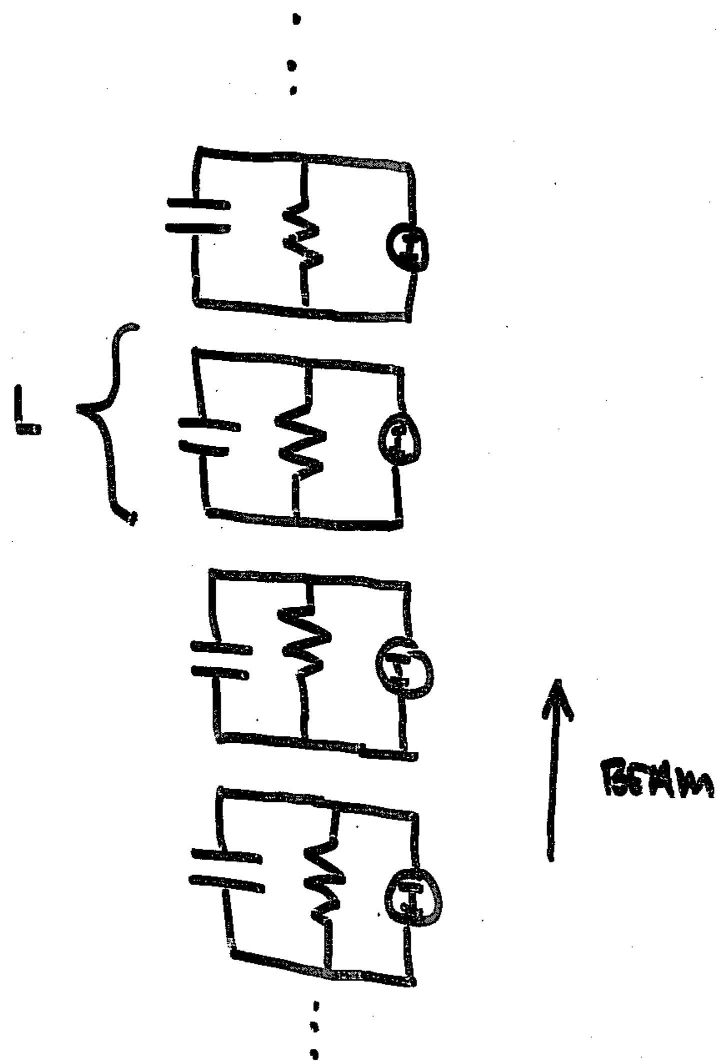
$$\Rightarrow E_1 = \frac{-R^*}{1 - i\omega C^+ R^*} I_1 \qquad Z^* \equiv -\frac{E_1}{I_1} = \frac{R^*}{1 - i\omega C^+ R^*}$$

RETURNING TO THE 1D FLUID EQUATIONS

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z}\lambda v = 0 \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = \frac{-qg}{4\pi\varepsilon_0 m}\frac{\partial \lambda}{\partial z} + \frac{qE_z}{m}$$

Let $\lambda = \lambda_0 + \lambda_1 \exp[i(kz - \omega t)]$

$u = v_0 + u_1 \exp[i(kz - \omega t)]$



CONTINUOUS LIMIT:

$R^* = R/L$ Resistance per unit length

$C^\dagger = CL$ C^{-1} per unit length

$E = \frac{\Delta V}{L}$ Average electric field

$$-i\omega\lambda_1 + ik\lambda_0 u_1 + ikv_0\lambda_1 = 0$$

$$-i\omega u_1 + ikv_0 u_1 + \underbrace{\frac{ikqq\lambda_1}{4\pi\varepsilon_0 m}}_{=\frac{ikc_s^2}{\lambda_0}\lambda_1} + \frac{q}{m} Z^* \underbrace{(\lambda_0 v_1 + v_0\lambda_1)}_{=I_1} = 0$$

$$\begin{bmatrix} \omega - kv_0 & -k\lambda_0 \\ -\frac{c_s^2 k}{\lambda_0} + \frac{iq}{m} Z^* v_0 & \omega - kv_0 + \frac{iq}{m} Z^* \lambda_0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ u_1 \end{bmatrix} = 0$$

THE DETERMINANT OF THE ABOVE MATRIX MUST VANISH:

$$(\omega - kv_0)^2 + \frac{iq}{m} Z^* \lambda_0(\omega - kv_0) - c_s^2 k^2 + \frac{iq}{m} Z^* \lambda_0 v_0 k = 0$$

$$\boxed{(\omega - kv_0)^2 - c_s^2 k^2 + \frac{iqZ^*\lambda_0\omega}{m} = 0}$$ (LAB FRAME)

Using a Galilean transformation, in the beam frame:

$$\omega' = \omega - kv_0$$ ' denotes beam frame

$$k' = k$$

$$\boxed{\omega'^2 - c_s^2 k'^2 + \frac{iqZ^*(\omega')\lambda_0(\omega' + k'v_0)}{m} = 0}$$ (BEAM FRAME)

NOTE $Z^*(\omega') = Z^*(\omega = \omega' + k'v_0)$

CASE I PURE RESISTIVE IMPEDANCE $Z^* = R^*$ (REAL)

$$\omega'^2 = \pm c_s k' \sqrt{1 - iR^* \frac{q\lambda_0}{m} \frac{(\omega' + k' V_0)}{c_s^2 k'^2}}$$

Using $c_s^2 = \frac{g q \lambda_0}{4\pi\epsilon_0 m}$ & $\frac{\omega'}{k'} \sim c_s << V_0$

$$\omega' = \pm c_s k' \sqrt{1 - iR^* \left(\frac{4\pi\epsilon_0}{g}\right) \frac{V_0}{k'}}$$

$$\simeq \pm \left[c_s k' - i \frac{c_s V_0}{2} \left(\frac{4\pi\epsilon_0}{g}\right) R^* \right]$$

Since $\lambda_1, E_1 \sim \exp\left[i(k'z' - \omega' t')\right]$

Choosing "+" ($\mathrm{Re}\,\omega > 0$) $\Rightarrow$ $z' = c_s t$ line of const phase $\Rightarrow$ Forward propagating

($\mathrm{Im}\,\omega' < 0$) $\Rightarrow$ $\lambda_1 \sim \exp\left[-\frac{c_s V_0}{2}\left(\frac{4\pi\epsilon_0}{g}\right) R^* t\right] \Rightarrow$ DECAYING PERTURBATION

CHOOSING "−"

($\mathrm{Re}\,\omega' < 0$) $\Rightarrow$ $z' = -c_s t'$ is line of constant phase

$\Rightarrow$ BACKWARD PROPAGATING

$$\Rightarrow \lambda_1 \sim \exp\left[\underbrace{+\frac{c_s V_0}{2}\left(\frac{4\pi\epsilon_0}{g}\right) R^* t}_{G}\right]$$

INSTABILITY!

$\lambda_1 \simeq \lambda_{10} \exp[G]$

$$G \equiv \Gamma_R t = \frac{4\pi\epsilon_0 c_s v_0 R^* t}{2g}$$

= LOGARITHMIC GAIN OF INSTABILITY $= \ln\left(\frac{\lambda_{1\,final}}{\lambda_{1\,initial}}\right)$

Now $t_{max} = \left\{ \min \begin{cases} l_b / c_s & \text{TRANSIT TIME FOR PERTURBATION TO TRAVEL FROM HEAD TO TAIL} \\ t_{residence} & \text{RESIDENCE TIME WITHIN ACCELERATOR} \end{cases} \right.$

If upper condition holds

$$G \sim \frac{v_0^2}{2}\left(\frac{4\pi\epsilon_0}{g}\right) R^* \Delta t$$

If lower condition holds

$$G \sim \sqrt{\lambda}$$

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$E = QV$

$$I \sim \frac{6\ \text{MJ}}{4\ \text{GeV}\ 200\ \text{ns}} \sim \frac{QV}{V\Delta t}$$

$$\sim 7.5\ \text{kA}$$

EXAMPLE:

FOR MATCHED BEAM IMPEDANCE

$$R^* = \frac{dV/ds}{I} \sim \frac{10^6\ \text{V/m}}{10\ \text{kA}} \sim 100\ \Omega/\text{m}$$

$V_0 \sim 0.2c$

$\Delta t \sim 200\ \text{ns}$

$$G \sim \frac{V_0^2}{2}\left(\frac{4\pi\epsilon_0}{g}\right) R^* \Delta t$$

~ 3.6

(AN EARLY CONCERN FOR HEAVY ION FUSION).

$R^* = 100\Omega/m$

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FOR ALL SIMULATIONS (p 11–15)

$V_0 = c/3$

$I = 3\,kA$

$l_b = 10\,m$

$\frac{r_b}{r_p} = 0.4$

$kT_\perp = kT_\parallel = 10\,keV$

a)

b)

c)

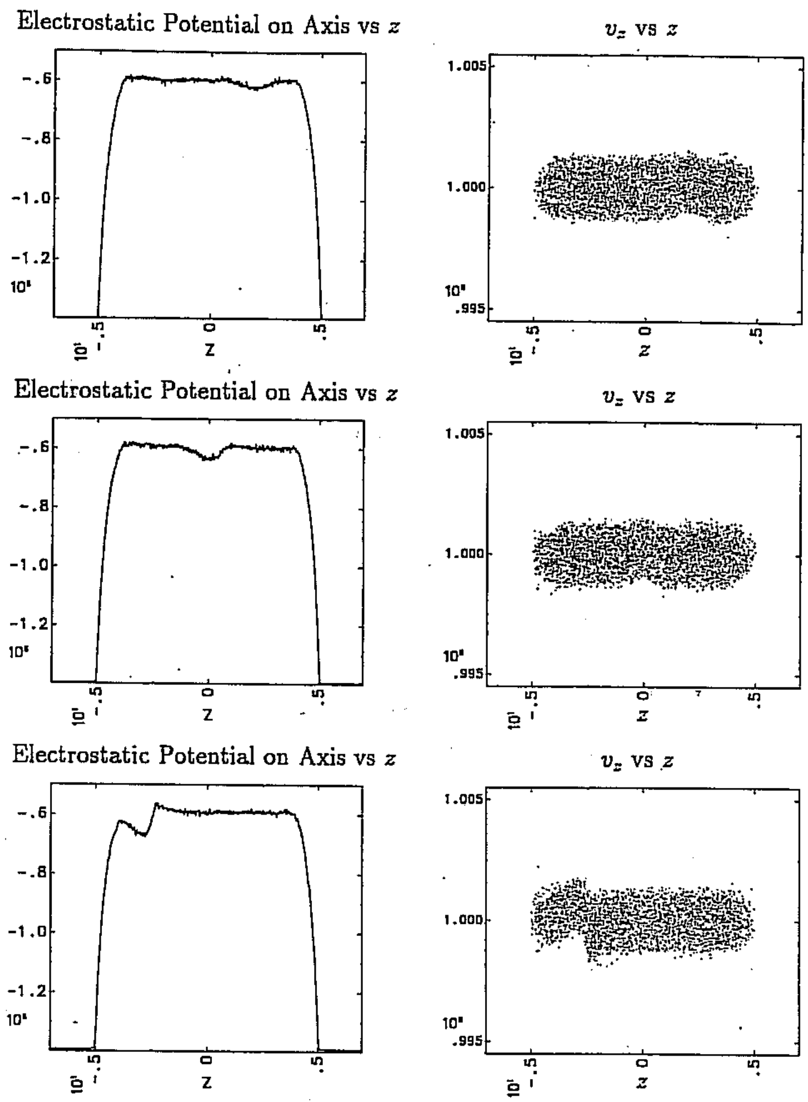


Figure 4.2: A simulation with 100 Ω/m resistance shows moderate growth. (a) 6.6 μs, (b) 10.9 μs, (c) 17.5 μs

from D. A. Callahan Miller, Ph.D. Thesis
U.C. Davis, 1994

$R^* = 100\ \Omega/m$ (12)

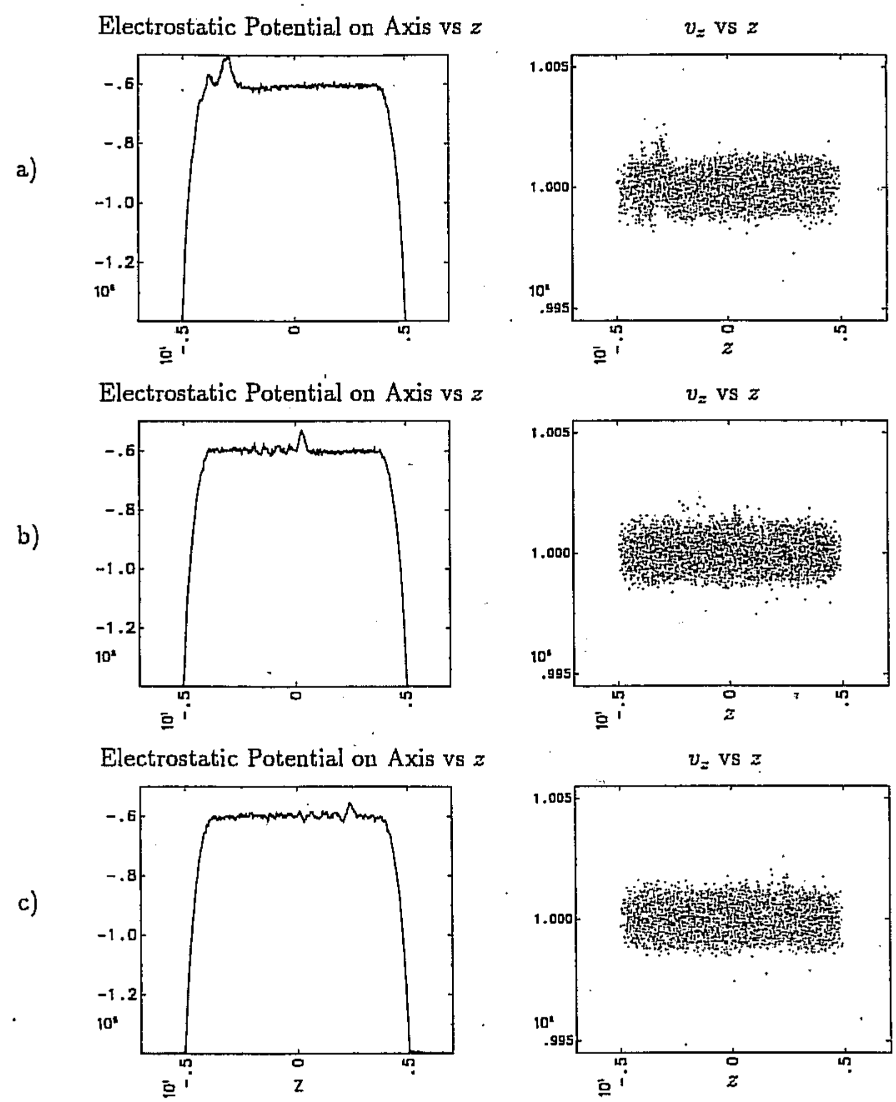


Figure 4.3: The perturbation reflects off the beam end and decays as it travels forward. (a) 28.4 μs, (b) 35.0 μs, (c) 39.4 μs

from D.A. Callahan Miller, Ph.D. Thesis
U.C. Davis, 1994
(FORWARD WAVE)

(13)

$R^* = 200\,\Omega/m$

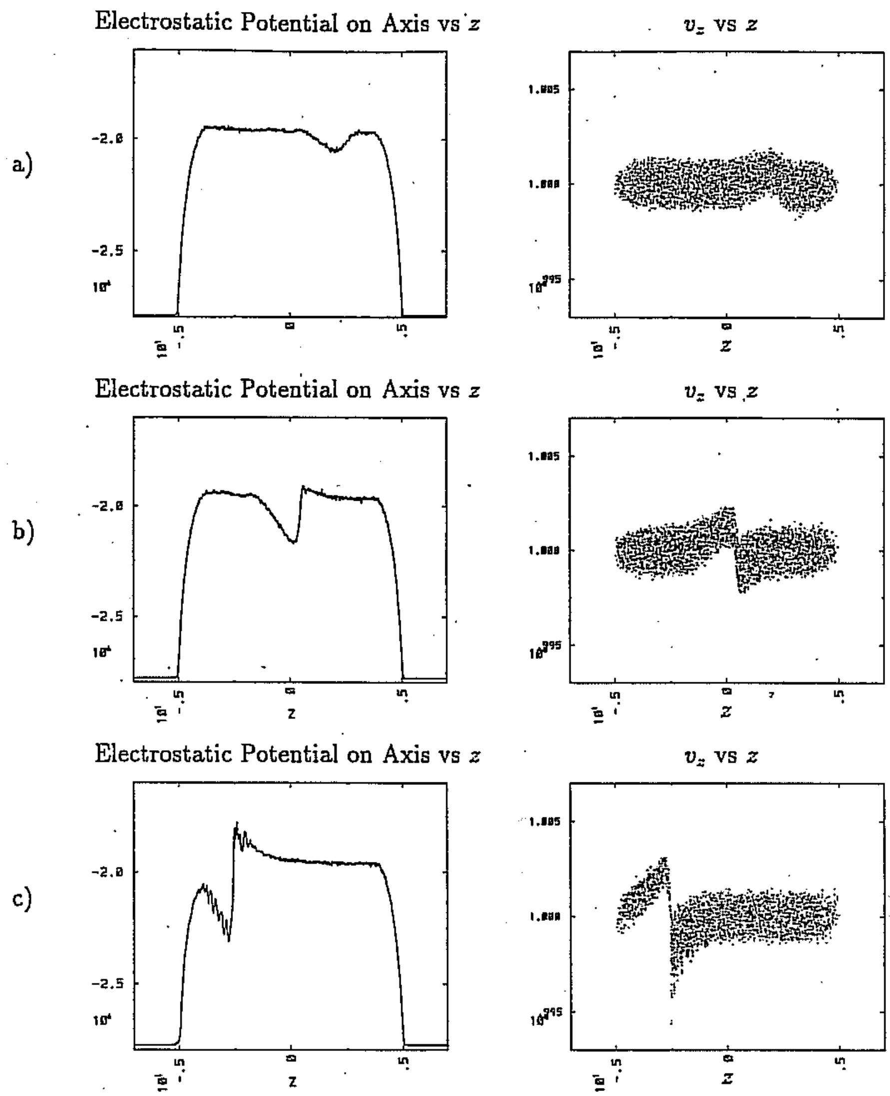


Figure 4.1: A simulation with 200 Ω/m resistance shows large amounts of growth. (a) 6.6 μs, (b) 10.9 μs, (c) 17.5 μs

from D.A. Callahan Miller, Ph.D. Thesis
U.C. Davis, 1994

CASE II RESISTIVE + CAPACITIVE IMPEDANCE

$$Z^* = \frac{R^*}{1 - i\omega C^+ R^*} = \frac{R^* + i\omega C^+ R^{*2}}{1 + \omega^2 C^{+2} R^{*2}}$$

GOING BACK TO PAGE 7:

IN LAB FRAME:

$$(\omega - kv_0)^2 - c_s^2 k^2 + \frac{iqR^* \lambda_0 \omega}{m(1 + \omega^2 C^{+2} R^{*2})} - \frac{q\omega^2 C^+ R^{*2} \lambda_0}{m(1 + \omega^2 C^{+2} R^{*2})} = 0$$

$$(\omega - kv_0)^2 - c_s^2 k^2 - \frac{4\pi\varepsilon_0}{g} \frac{\omega^2 C R^{*2} c_s^2}{(1 + \omega^2 C^2 R^{*2})} + \frac{i 4\pi\varepsilon_0 c_s^2 R_x^* \omega}{g(1 + \omega^2 C^2 R^{*2})}$$

IN BEAM FRAME:

$$\omega'^2 - c_s^2 k^2 - \frac{4\pi\varepsilon_0}{g} \frac{(\omega' + kv_0)^2 C R^{*2} c_s^2}{(1 + (\omega' + kv_0)^2 C^2 R^2)} + \frac{i 4\pi\varepsilon_0 c_s^2 R_x^2 (\omega' + kv_0)}{g(1 + \omega^2 C^2 R^{*2})}$$

So if one takes limit $C \to \infty$ the final two terms tend to zero. Thus capacitance has reduced the instability growth rate.

$$\omega'^2 - c_s^2 k'^2 - \frac{2\Gamma_R (c_s/v_0)(\omega' + k'v_0)^2 C^\dagger R^*}{(1+(\omega'+k'v_0)^2 C^{\dagger 2} R^{*2})} + \frac{2\Gamma_R (c_s/v_0)(\omega'+k'v_0)}{(1+(\omega'+k'v_0)^2 C^{\dagger 2} R^{*2})} = 0$$

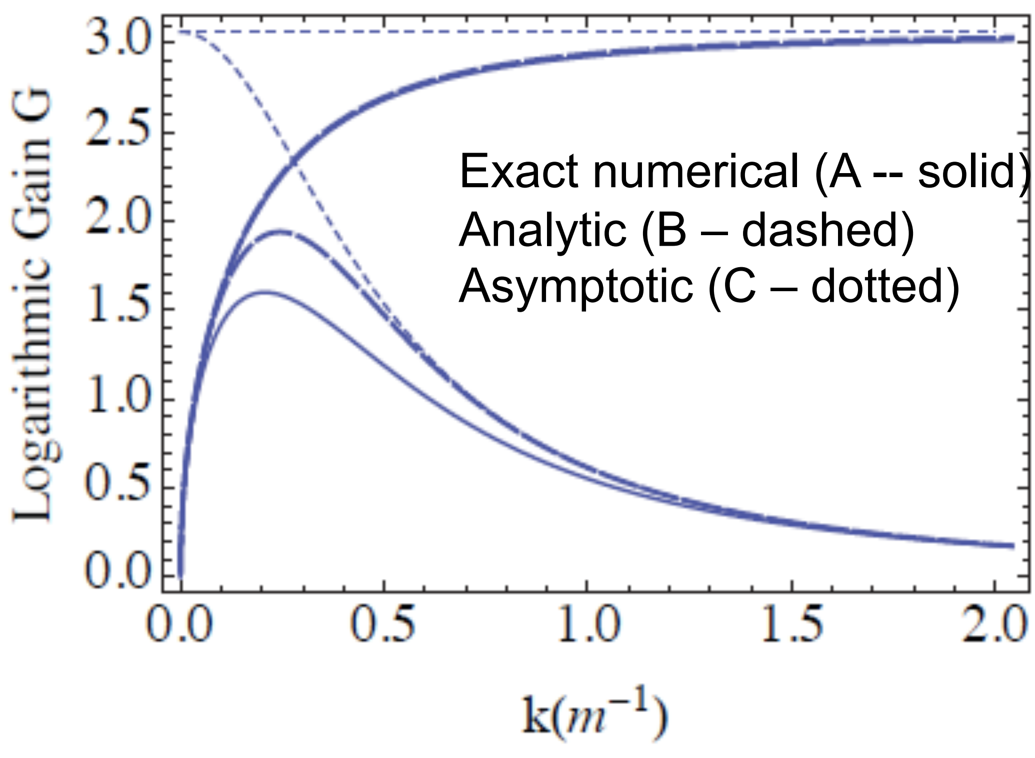
For $2(c_s/v_0)\Gamma_R R^* C^\dagger << 1$: (A)

$$\omega' = \pm c_s k' \sqrt{1 - i\frac{2\Gamma_R}{c_s k' \left(1+(k'v_0 R^* C^\dagger)^2\right)}}$$

$$= \pm c_s k' \left(1 + \left(\frac{2\Gamma_R}{c_s k' \left(1+(k'v_0 R^* C^\dagger)^2\right)}\right)^2\right)^{1/4} \times \quad \text{(B)}$$

$$\times \left(\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{2\Gamma_R}{c_s k' \left(1+(k'v_0 R^* C^\dagger)^2\right)}\right)\right] - i\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{2\Gamma_R}{c_s k' \left(1+(k'v_0 R^* C^\dagger)^2\right)}\right)\right]\right)$$

$$\omega' \simeq \pm c_s k' \mp i\frac{\Gamma_R}{1+(k'v_0 R^* C^\dagger)^2} \qquad \text{for} \quad \frac{2\Gamma_R}{c_s k'(1+(k'v_0 R^* C^\dagger)^2)} << 1 \quad \text{(C)}$$



Logarithmic Gain G of the longitudinal instability as a function of perturbation wavenumber k, for R = 100 /m, $C^\dagger = 0$ (upper curves) and $C^\dagger = 2 \times 10^{-10}$ F-m (lower curves), after a growth time corresponding to l_b/c_s, where l_b = 10 m and cs = 4.9 x10^5 m

$R^*C^* = 2 \times 10^{-8}$ s
$R^* = 100\Omega/m$

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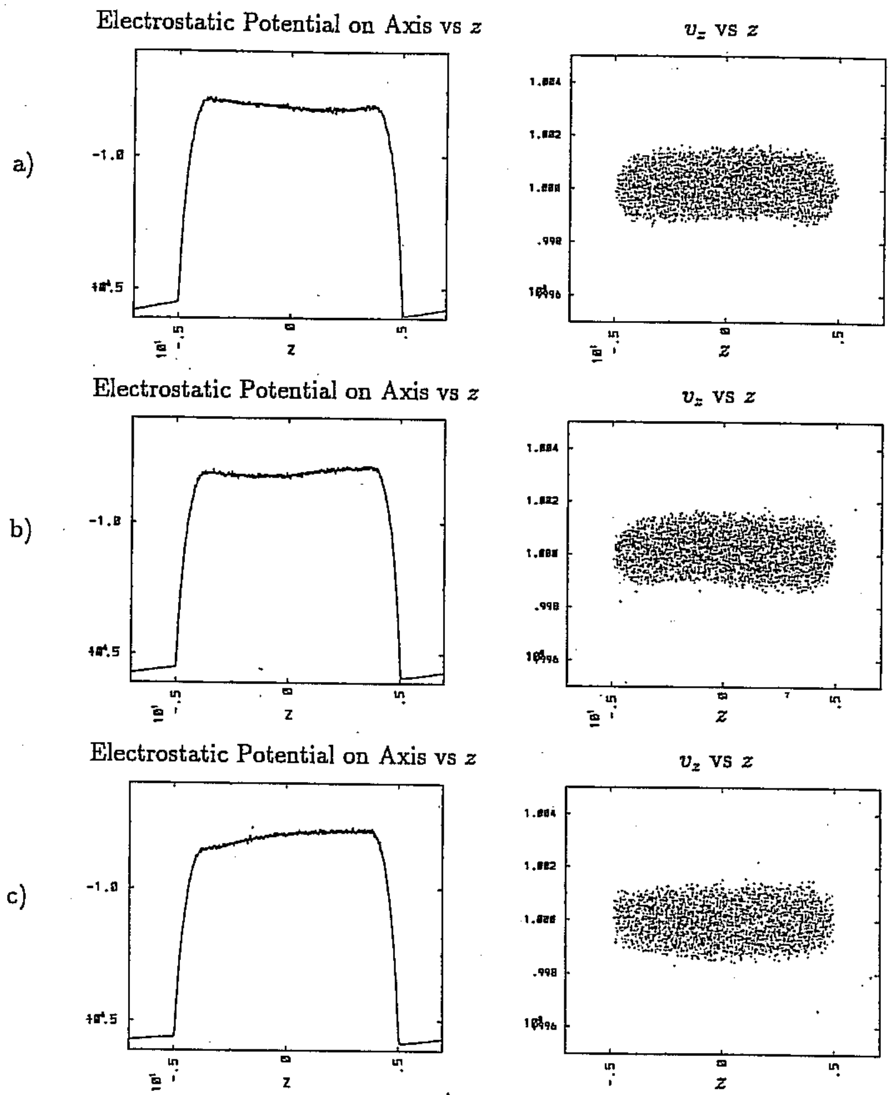


Figure 4.6: When capacitance is added to the system, a larger perturbation is launched, but little growth occurs (a) 6.6 μs, (b) 10.9 μs, (c) 17.5 μs

from D.A. Callahan Miller, Ph.D. Thesis, U.C. Davis, 1994

Summary of Longitudinal Instability

"Resistive Wall" or "Longitudinal" Instability has potential to degrade longitudinal emittance in high current accelerators.

However, capacitance (e.g. from accelerating gaps) decreases growth can mitigate instability,

Not Discussed:

1. Longitudinal temperature damps instability (c.f. Reiser 6.3.3)

2. Feedback has been provided to control instability if needed

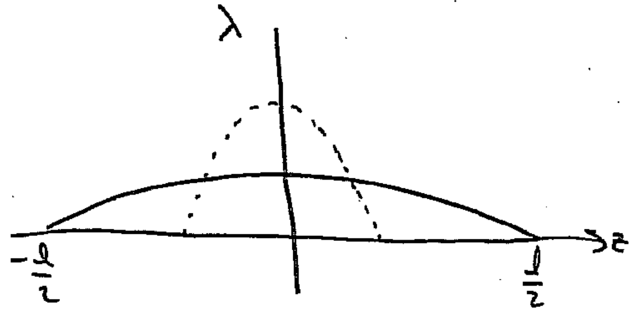
DRIFT COMPRESSION

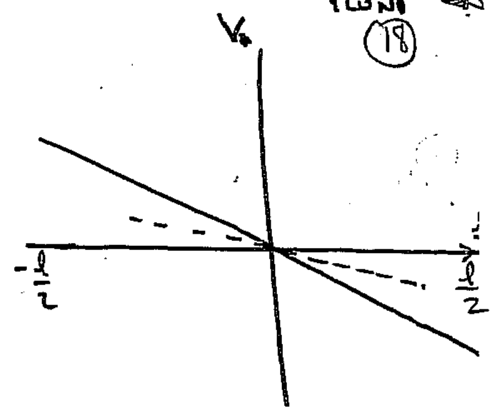
OBJECTS:

APPLY A HEAD-TO-TAIL VELOCITY TILT TO INCREASE CURRENT BY DECREASING PULSE DURATION

DURING COMPRESSION "EARS" ARE NOT REQUIRED

AT END OF DRIFT COMPRESSION, VELOCITY "TILT" SHOULD BE MINIMIZED, SO THAT CHROMATIC ABERRATIONS IN FINAL FOCUS ARE MINIMIZED.





$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z}\lambda V = 0$$ CONTINUITY EQUATION

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = \frac{-q g}{m\, 4\pi\epsilon_0}\frac{\partial \lambda}{\partial z}$$ MOMENTUM EQUATION

LET $\lambda = \lambda_0(t)\left(1 - \frac{4z^2}{l^2(t)}\right)$ ← PARABOLIC LINE CHARGE PROFILE

$V = -\Delta V(t)\,\frac{z}{l(t)}$ ← LINEAR VELOCITY PROFILE

① Mass conservation:

$$Q_c = \int_{-l/2}^{l/2} \lambda\, dz = \lambda_0 \int_{-\frac{l}{2}}^{l/2}\left(1 - \frac{4z^2}{l^2}\right) dz = \frac{2}{3}\lambda_0 l = \text{constant}$$

(but $\lambda_0 = \lambda_0(t)$ & $l = l(t)$)

CALCULATING PARTIAL DERIVATIVES:

$$\frac{\partial \lambda}{\partial t} = \dot{\lambda}_0\left(1-\frac{4z^2}{\ell^2}\right) + 2\lambda_0\left(\frac{4z^2}{\ell^3}\right)\dot{\ell}$$

$$\frac{\partial \lambda}{\partial z} = \frac{-8z}{\ell^2}\lambda_0$$

$$\frac{\partial V}{\partial t} = -\Delta\dot{V}\left(\frac{z}{\ell}\right) + \frac{\Delta V\, z}{\ell^2}\dot{\ell}$$

FROM DEFINITION OF ΔV & $\dot{\ell}$:

$\Delta V = -\dot{\ell}$

$$\frac{\partial V}{\partial z} = -\frac{\Delta V}{\ell}$$

(2) CONTINUITY EQUATION $\Rightarrow$

$$\begin{cases} \dfrac{8z^2\lambda_0}{l^3}\left(\Delta v + \dot{\ell}\right) = 0 \\ \left(1-\dfrac{4z^2}{\ell^2}\right)\left(\dot{\lambda}_0 - \dfrac{\Delta V \lambda_0}{\ell}\right) = 0 \end{cases}$$

(3) MOMENTUM EQUATION $\Rightarrow$

$$\left(\frac{z}{\ell}\right)\left[-\Delta\dot{V} + \frac{\dot{\ell}\,\Delta V}{\ell} + \frac{\Delta V^2}{\ell} + \frac{8qg}{m\,4\pi\epsilon_0}\frac{\lambda_0}{\ell}\right] = 0$$

(1) & (2) $\Rightarrow$

$$\frac{\dot{\lambda}_0}{\lambda_0} = \frac{\Delta V}{\ell} = \frac{-\dot{\ell}}{\ell} \qquad (4)$$

(3) & (4) $\Rightarrow$

$$\boxed{\ddot{\ell} - \frac{12qg}{4\pi\epsilon_0 m}\frac{Q_c}{\ell^2} = 0}$$

where $Q_c = \frac{2}{3}\lambda_0 \ell = \text{const.}$

$\equiv$ CHARGE IN BUNCH (NOT PERVEANCE)

LONGITUDINAL "ENVELOPE" EQUATION
(WITHOUT EMITTANCE)

MULTIPLY BY $\dot{l}$ & INTEGRATE:

$$\frac{\dot{l}^2}{2} + \frac{12qg}{4\pi\epsilon_0 m}\frac{Q_c}{l} = \underbrace{\cancel{\frac{\dot{l}_f^2}{2}}}_{0} + \frac{12qg}{4\pi\epsilon_0 m}\frac{Q_c}{l_f}$$

HERE SUBSCRIPT "f" = "final"

& SUBSCRIPT "o" = original or initial

$$\Rightarrow \quad \dot{l}_o = \sqrt{\frac{16qg}{4\pi\epsilon_0 m}\lambda_f\left[1-\frac{l_f}{l_o}\right]}$$

Now $Q_F = \dfrac{\lambda_f}{4\pi\epsilon_0 V_f}$ = FINAL PERVEANCE AT CENTER OF PARABOLIC PULSE

(NOTE $Q_c \equiv \frac{2}{3}\lambda_o l$ = CHARGE

whereas Q_F ≡ PERVEANCE (DIMENSIONLESS)

$$C = \text{COMPRESSION RATIO} = \frac{l_o}{l_f}$$

$$\frac{\Delta V}{V_o} = \text{velocity tilt} = \frac{|\dot{l}|}{V_o}$$

$$\rightarrow \quad \boxed{\frac{\Delta V}{V} = \sqrt{8gQ_f\left[1-\frac{1}{C}\right]}}$$

for $Q_f = 10^{-4}$

$g = 1.1$

$C = 20$

$$\Rightarrow \quad \frac{\Delta V}{V} = 0.029$$

$$\boxed{\text{DRIFT LENGTH} \cong \frac{l}{\Delta V}V_o = \frac{l}{\Delta V/V} = 345\text{ m} \quad \text{for } l = 10\text{ m}}$$

Vlasov - equation for a drifting beam:

$$\frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + x''\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z} + z''\frac{\partial f}{\partial z'} = 0$$

Let $\tilde{f}(z, z', s) \equiv \iiiint f \, dx \, dx' \, dy \, dy'$

INTEGRATING VLASOV EQUATION: If $z'' \neq f(x, x', y, y')$:

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \iiint x' \underbrace{\int \frac{\partial f}{\partial x} dx}_{= f\big|_{-\infty}^{\infty} = 0} dx' \, dy \, dy' + \cdots \quad z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0} \qquad \text{1 D Vlasov}$$

Now let $\lambda \equiv q\int \tilde{f} \, dz'$; $\lambda \overline{z'} = q\int \tilde{f} z' \, dz'$; $q\lambda \overline{z'^2} \equiv \int \tilde{f} z'^2 \, dz'$

Also, let $\Delta z'^2 \equiv \overline{z'^2} - (\overline{z'})^2$

FLUID EQUATIONS

INTEGRATING 1D VLASOV OVER z':

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \overline{z'}) = 0} \qquad \text{(CONTINUITY EQUATION)}$$

MULTIPLYING BY z' & INTEGRATING VLASOV OVER z':

$$\frac{\partial}{\partial s}\lambda \overline{z'} + \frac{\partial}{\partial z}\lambda \overline{z'^2} - \lambda z'' = 0$$

DIVIDING BY λ, USING CONTINUITY EQUATION & DEFINITION OF $\Delta z'^2$:

$$\boxed{\underbrace{\frac{\partial \overline{z'}}{\partial s} + \overline{z'}\frac{\partial}{\partial z}\overline{z'}}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2)}_{\text{PRESSURE TERM}} = \underbrace{\overline{z''}}_{\text{FORCE}}} \qquad \text{(MOMENTUM EQUATION)}$$

LONGITUDINAL ENVELOPE EQUATION

$$\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0$$

Q_c = total charge in bunch

If $z'' = -K(s) z + \frac{qg}{4\pi\epsilon_0 m v^2}\left(\frac{12 Q_c}{L^3}\right) z$

$$\Rightarrow \frac{\partial}{\partial s}\langle z^2\rangle = 2\langle z z'\rangle$$

$$\frac{\partial}{\partial s}\langle z z'\rangle = \langle z'^2\rangle + \frac{qg}{4\pi\epsilon_0 m v^2}\left(\frac{12 Q_c}{L^3}\right)\langle z^2\rangle - K(s)\langle z^2\rangle$$

$$\frac{\partial}{\partial s}\langle z'^2\rangle = 2\left(\frac{qg}{4\pi\epsilon_0 m v^2}\right)\left(\frac{12 Q_c}{L^3}\right)\langle z z'\rangle - 2K(s)\langle z z'\rangle$$

NOTE $\langle z^2\rangle = \frac{1}{Q_c}\int_{-\infty}^{\infty}\int_{-L/2}^{L/2} z^2 f(z,z')\, dz\, dz' = \frac{1}{20} L^2$

$$\varepsilon_z^2 = 25\left[\langle z^2\rangle\langle z'^2\rangle - \langle z z'\rangle^2\right]$$

$$\Rightarrow \frac{d^2 L}{ds^2} = \frac{16\,\varepsilon_z^2}{L^3} + \frac{12\, g q Q_c}{4\pi\epsilon_0 m v^2 L^2} - K(s) L$$

Let $r_z = L/2$

$$\Rightarrow \boxed{\frac{d^2 r_z}{ds^2} = \frac{\varepsilon_z^2}{r_z^3} + \frac{3}{2}\frac{g q Q_c}{4\pi\epsilon_0 m v^2}\frac{1}{r_z^2} - K(s) r_z}$$

If we regard the envelope radii $r_x,\ r_y$ as specified functions of *s*, then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$\boxed{\begin{aligned} x''(s) + \kappa_x^{\text{eff}}(s)x(s) &= 0 \\ y''(s) + \kappa_y^{\text{eff}}(s)y(s) &= 0 \\ \kappa_x^{\text{eff}}(s) = \kappa_x(s) &- \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} \\ \kappa_y^{\text{eff}}(s) = \kappa_y(s) &- \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)} \end{aligned}}$$

Suggests Procedure:

- Calculate Courant-Snyder invariants under assumptions made
- Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
  - Nontrivial step: guess and show that it works

Resulting distribution will be an equilibrium that does not evolve in *s* in 4D phase-space, but lower-dimensional phase-space projections can evolve in *s*

Self – consistent longitudinal distribution

Recall Hill's equation: (From Steve Lund's notes on "Transverse equilibrium distributions," p. 20 -26.)

$z'' + K(s)\, z = 0$

The Courant-Snyder invariant Cz can be written:

$$C_z = \left(\frac{z}{r_z}\right)^2 + \left(\frac{r_z z' - r_z' z}{\varepsilon_z}\right)^2$$

= constant along a particle trajectory

At each s, particle lies on ellipse of constant area $\pi\varepsilon_z$.

Along each trajectory: $\frac{dC_z}{ds} = 0$

$$\frac{df}{ds} = \frac{df}{dC_z}\frac{dC_z}{ds} = 0$$

so $f(C_z)$ is a solution of the Vlasov equation.

BUT $\lambda = \int f(z, z', s)\, dz'$ must be of the form

$$\lambda = (a_0 + b_0 z^2) \quad \Rightarrow \quad E_z = -g\frac{\partial \lambda}{\partial z} \sim z$$

So what $f(C_z)$ yields $\lambda = (a_0 + b_0 z^2)$?

Answer: $\boxed{f(C_z) = \frac{3N}{2\pi\varepsilon_z}\sqrt{1 - C_z}}$

SELF-CONSISTENT LONGITUDINAL DISTRIBUTION:

NEUFFER DISTRIBUTION — D. NEUFFER, PARTICLE ACCELERATORS, Vol 11, p 23 (1980)

RETURNING TO THE 1D VLASOV EQUATION:

$$\frac{\partial f}{\partial s} + z' \frac{\partial f}{\partial z} + z'' \frac{\partial f}{\partial z'} = 0$$

If $z'' = -A \dfrac{\partial \lambda}{\partial z} - K(s) z$

THEN,

$$f(z, z', s) = \frac{3N}{2\pi \varepsilon_z} \sqrt{1 - \frac{z^2}{r_z^2} - \frac{r_z^2 (z' - r_z' z / r_z)^2}{\varepsilon_z^2}}$$

for $-r_z < z < r_z$

& $\dfrac{r_z' z}{r_z} - \dfrac{\varepsilon_z}{r_z}\sqrt{1 - \dfrac{z^2}{r_z^2}} \leq z' \leq \dfrac{r_z' z}{r_z} + \dfrac{\varepsilon_z}{r_z}\sqrt{1 - \dfrac{z^2}{r_z^2}}$

is a solution to the 1D Vlasov equation.

HERE $\varepsilon_z^2 = 25\,(\langle z^2\rangle\langle z'^2\rangle - \langle z z'\rangle^2] = $ CONSTANT

N = total number of particles in bunch

r_z = hard edge of bunch

NOTE THAT $\lambda(z) = \dfrac{3}{4}\dfrac{N}{r_z}\left(1 - \dfrac{z^2}{r_z^2}\right) = \displaystyle\int_{-r_z}^{r_z} f(z, z', s)\, dz'$

$\Rightarrow \dfrac{\partial \lambda}{\partial z} \propto z \Rightarrow$ LINEAR SPACE CHARGE FIELD

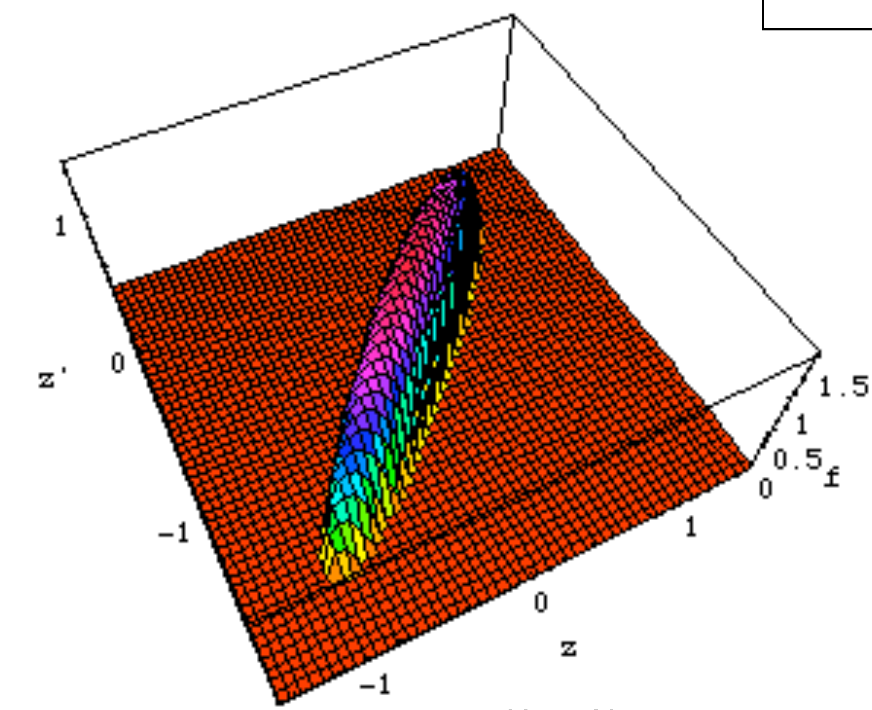
Neuffer Distribution Function

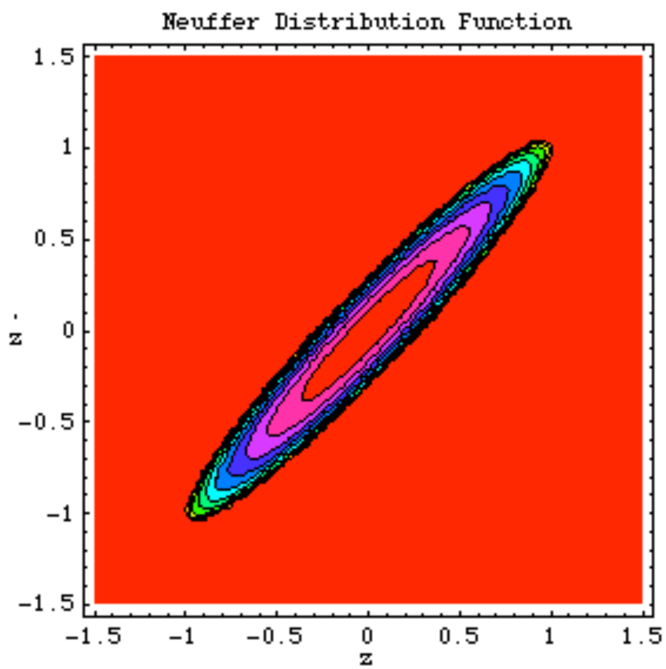
$$f[z,z'] = \frac{3N}{2\pi\varepsilon_z}\sqrt{1-\frac{z^2}{r_z^2}-\frac{r_z^2(z'-r_z'z/r_z)^2}{\varepsilon_z^2}}$$

for:

$$-r_z \le z \le r_z$$

$$\frac{r_z'z}{r_z}-\frac{\varepsilon_z}{r_z}\sqrt{1-\frac{z^2}{r_z^2}} \le z' \le \frac{r_z'z}{r_z}+\frac{\varepsilon_z}{r_z}\sqrt{1-\frac{z^2}{r_z^2}}$$





Here $N=r_z=r_z'=1$; $\varepsilon_z=0.3$

Summary

1D VLASOV EQUATION

$$\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0$$

& g-factor model

$$z'' = \frac{-g}{4\pi\epsilon_0 m V_0^2} \frac{\partial \lambda}{\partial z}$$

Leads to fluid equations:

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \bar{z}') = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2) + \frac{c_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda}{\partial z} = 0$$

⇒ SPACE CHARGE WAVES

↳ LONGITUDINAL OR RESISTIVE WALL INSTABILITY

⇒ SPACE CHARGE RAREFACTION WAVES

⇒ PARABOLIC BUNCH COMPRESSION $\frac{\partial \lambda}{\partial z} \propto z$

VLASOV EQUATION ALSO ⇒ ENVELOPE EQUATION

$$\frac{d^2 r_z}{ds^2} = \frac{\epsilon_z^2}{r_z^3} + \frac{3}{2}\frac{g q Q_c}{4\pi\epsilon_0 m V^2}\frac{1}{r_z^2} - K(s) r_z$$

KINETIC SOLUTION TO VLASOV EQUATION SATISFYING rms ENVELOPE EQUATION IS "NEUFFER DISTRIBUTION" (ANALOGOUS TO KV).

$$f(z,z') = \frac{3N}{2\pi\epsilon_z}\sqrt{1 - \frac{z^2}{r_z^2} - \frac{r_z^2 (z' - r_z' z / r_z)^2}{\epsilon_z^2}}$$

Summary

1D VLASOV EQUATION

& g-factor model

$$\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0 \qquad z'' = \frac{-g}{4\pi\epsilon_0 m V_0^2} \frac{\partial \lambda}{\partial z}$$

Leads to fluid equations:

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \bar{z}') = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2) + \frac{c_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda}{\partial z} = 0$$

$\Rightarrow$ SPACE CHARGE WAVES

$\hookrightarrow$ LONGITUDINAL OR RESISTIVE WALL INSTABILITY

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$$\frac{d^2 r_z}{ds^2} = \frac{\epsilon_z^2}{r_z^3} + \frac{3}{2}\frac{g q Q_c}{4\pi\epsilon_0 m V^2}\frac{1}{r_z^2} - K(s) r_z$$

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