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II. Envelope Equations

Paraxial Ray Equation

Envelope equations for axially
symmetric beams

Cartesian equation of motion

Envelope equations for elliptically
symmetric beams

Roadmap:

Single particle equation with Lorentz force
 $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

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Make use of:

1. Paraxial (near-axis) approximation
(Small r and r')
2. Conservation of canonical angular momentum
3. Axisymmetry $f(r,z)$

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Paraxial Ray Equation for Single Particle

Next take statistical averages over the distribution function

⇒ Moment equations

Express some of the moments in terms of the rms radius and emittance

⇒ Envelope equations (axi-symmetric case)

Some focusing systems have quadrupolar symmetry

Rederive envelope equations in cartesian coordinates (x,y,z) rather than radial (r,z)

START WITH NEWTON'S EQUATION WITH THE LORENTZ FORCES:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

In cartesian coordinates:

$$\frac{d}{dt}(\gamma m \dot{x}) = \gamma m \ddot{x} + \dot{\gamma} m \dot{x} = q(E_x + \dot{y} B_z - \dot{z} B_y)$$

$$\frac{d}{dt}(\gamma m \dot{y}) = \gamma m \ddot{y} + \dot{\gamma} m \dot{y} = q(E_y + \dot{z} B_x - \dot{x} B_z)$$

$$\frac{d}{dt}(\gamma m \dot{z}) = \gamma m \ddot{z} + \dot{\gamma} m \dot{z} = q(E_z + \dot{x} B_y - \dot{y} B_x)$$

In cylindrical coordinates: (use $\frac{d\hat{e}_\theta}{dt} = \hat{e}_\theta \dot{\theta}$; $\frac{d\hat{e}_\phi}{dt} = -\hat{e}_\phi \dot{\theta}$)
(see next page)

$$\frac{d}{dt}(\gamma m r \dot{r}) - \gamma m r \dot{\theta}^2 = q(E_r + r \dot{\theta} B_z - \dot{z} B_\theta) \quad (I)$$

$$\frac{1}{r} \frac{d}{dt}(\gamma m r^2 \dot{\theta}) = q(E_\theta + \dot{z} B_r - \dot{r} B_z) \quad (II)$$

$$\frac{d}{dt}(\gamma m \dot{z}) = q(E_z + \dot{r} B_\theta - \dot{\theta} B_r) \quad (III)$$

When $\frac{\partial A}{\partial t} = 0$:

$$\underline{E} = -\underline{\nabla} \phi = \hat{e}_r \left[-\frac{\partial \phi}{\partial r} - \frac{\partial A_\theta}{\partial t} \right] + \hat{e}_\theta \left[-\frac{\partial \phi}{\partial \theta} \right] + \hat{e}_z \left[-\frac{\partial \phi}{\partial z} - \frac{\partial A_z}{\partial t} \right]$$

$$\underline{B} = \underline{\nabla} \times \underline{A} = \hat{e}_r \left[-\frac{\partial}{\partial z} A_\theta \right] + \hat{e}_\theta \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right] + \hat{e}_z \left[\frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \right]$$

RHS of II multiplied by r:

$$q r (E_\theta + \dot{z} B_r - \dot{r} B_z) = q \left(-\frac{\partial r A_\theta}{\partial t} - \dot{z} \frac{\partial r A_\theta}{\partial z} - \dot{r} \frac{\partial}{\partial r} (r A_\theta) \right)$$

$$= -q \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \underline{\nabla} \right] (r A_\theta)$$

$$= -q \frac{d}{dt} (r A_\theta)$$

(IV)

$$\text{Eqn II} \Rightarrow \frac{d}{dt} (-\gamma m r^2 \dot{\theta} + q r A_\theta) = 0$$

TO CALCULATE THE RATE OF CHANGE OF THE MOMENTUM \underline{p} IN CYLINDRICAL COORDINATES, WE MUST TAKE INTO ACCOUNT THAT THE UNIT VECTOR CHANGES DIRECTION AS THE PARTICLE MOVES:

$$\text{LET } \underline{p} = p_r \hat{e}_r + p_\theta \hat{e}_\theta + p_z \hat{e}_z = \gamma m \underline{v}$$

$$\text{WHEN } p_r = \gamma m \dot{r}$$

$$p_\theta = \gamma m r \dot{\theta}$$

$$p_z = \gamma m \dot{z}$$

NOTE: ON THIS PAGE $p_\theta^* \equiv \theta$ -component of mechanical momentum
NOT TO BE CONFUSED WITH
 $p_\theta \equiv \gamma m r^2 \dot{\theta} + q r A_\theta = \theta$ -component of CANONICAL ANGULAR momentum

$$\text{SO } \frac{d\underline{p}}{dt} = \dot{p}_r \hat{e}_r + p_r \dot{\hat{e}}_r + \dot{p}_\theta \hat{e}_\theta + p_\theta \dot{\hat{e}}_\theta + \dot{p}_z \hat{e}_z$$

$$\Rightarrow \frac{d\underline{p}}{dt} = (\dot{p}_r - p_\theta \dot{\theta}) \hat{e}_r + (p_r \dot{\theta} + \dot{p}_\theta) \hat{e}_\theta + \dot{p}_z \hat{e}_z$$

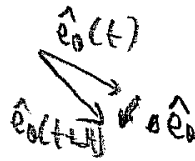
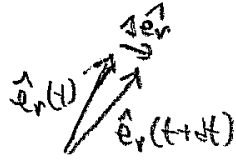
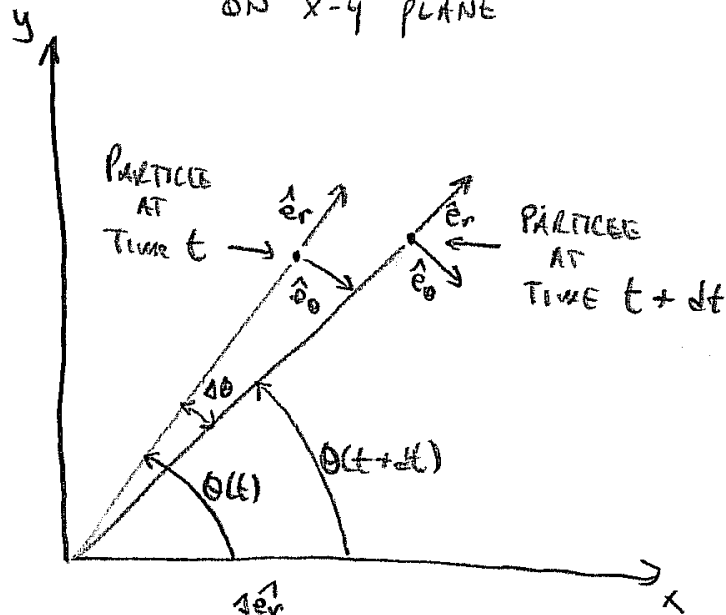
WHERE WE HAVE USED:

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta} \quad \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$$

$$\begin{aligned} \Rightarrow \frac{d\underline{p}}{dt} &= \left[\frac{d}{dt} (\gamma m \dot{r}) - (\gamma m r \dot{\theta}^2) \right] \hat{e}_r \\ &+ \left[\gamma m \dot{r} \dot{\theta} + \frac{d}{dt} (\gamma m r \dot{\theta}) \right] \hat{e}_\theta + \frac{d}{dt} (\gamma m \dot{z}) \hat{e}_z \\ &= \frac{1}{r} \underbrace{\frac{d}{dt} (\gamma m r^2 \dot{\theta})}_{\text{MECHANICAL ANGULAR MOMENTUM}} \hat{e}_\theta + \frac{d}{dt} (\gamma m \dot{z}) \hat{e}_z \end{aligned}$$

MECHANICAL ANGULAR MOMENTUM

PROJECTION OF PARTICLE POSITION AT TIMES t & $t+dt$ ON X-Y PLANE (5)



$$\Delta \hat{e}_r = \hat{e}_\theta \Delta \theta$$

$$\Delta \hat{e}_\theta = -\hat{e}_r \Delta \theta$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta}$$

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$$

CONSERVATION OF CANONICAL ANGULAR MOMENTUM

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DEFINE $p_\theta = \gamma m r^2 \dot{\theta} + q r A_\theta$

$$\frac{d}{dt} p_\theta = 0$$

(CONSERVATION OF
CANONICAL ANGULAR MOMENTUM)

NOTE THAT THE FLUX ENCLOSED BY A CIRCLE OF RADIUS r

$$\psi = \int \underline{B} \cdot \underline{dA} = \int (\nabla \times \underline{A}) \cdot \underline{dA} = \oint \underline{A} \cdot \underline{d\ell} = 2\pi r A_\theta$$

$$p_\theta = \gamma m r^2 \dot{\theta} + \frac{q\psi}{2\pi}$$

IS CONSERVED ALONG AN ORBIT
IN AXISYMMETRIC GEOMETRIES

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"EXTERNAL" ELECTRIC & MAGNETIC FIELD WITH

AZIMUTHAL SYMMETRY ($\frac{\partial}{\partial \phi} = 0$) (REIFER SECTION 5.3)

CONSIDER THE FIELD $\underline{E}_{\text{ext}}$ & $\underline{B}_{\text{ext}}$ CREATED BY EXTERNAL SOURCES:

(TIME STEADY, VACUUM FIELDS):

$$\nabla \times \underline{B}_{\text{ext}} = 0 \quad \nabla \times \underline{E}_{\text{ext}} = 0 \quad (\Rightarrow E, B \sim \nabla \phi)$$

$$\nabla \cdot \underline{B}_{\text{ext}} = 0 \quad \nabla \cdot \underline{E}_{\text{ext}} = 0 \quad \Rightarrow \nabla^2 \phi = 0$$

In cylindrical coordinates:

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2}$$

$$\text{Let } \phi(r, z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z) r^{2\nu} = f_0 + f_2 r^2 + f_4 r^4 + \dots$$

$$\nabla^2 \phi = 0 \quad \Rightarrow \sum_{\nu=1}^{\infty} (2\nu)^2 f_{2\nu} r^{2\nu-2} + \sum_{\nu=0}^{\infty} f_{2\nu}'' r^{2\nu} = 0$$

$$\Rightarrow f_0''(z) = -4 f_2$$

$$f_2''(z) = -16 f_4$$

$$f_4''(z) = -36 f_6$$

$$\Rightarrow f_{2\nu} = \frac{(-1)^\nu}{(\nu!)^2 2^{2\nu}} \frac{\partial^{2\nu} f(z)}{\partial z^{2\nu}}$$

$$\Rightarrow \phi(r, z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} f(z)}{\partial z^{2\nu}} \left(\frac{r}{z} \right)^{2\nu} = f_0(z) - \frac{1}{4} f_0''(z) r^2 + \frac{1}{64} f_0''''(z) r^4 + \dots$$

Let $B_z(0, z) = B(z)$ & LET $\phi(0, z) = V(z) = f_0(z)$

$$B_z(r, z) = \frac{-\partial \phi(r, z)}{\partial z} = f_0'(z) - \frac{1}{4} f_0''''(z) r^2 + \frac{1}{64} f_0''''''(z) r^4 + \dots$$

$$= B(z) - \frac{r^2}{4} \frac{\partial^2 B(z)}{\partial z^2} + \frac{r^4}{64} \frac{\partial^4 B(z)}{\partial z^4} + \dots$$

$$B_r(r, z) = \frac{-\partial \phi(r, z)}{\partial r} = -\frac{r}{z} \frac{\partial B(z)}{\partial z} + \frac{r^3}{16} \frac{\partial^3 B}{\partial z^3} + \dots$$

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Similarly, for the electric field define

$$V(z) = \Phi(0, z) = f_0(z)$$

$$\Rightarrow \Phi(r, z) = V(z) - \frac{r^2}{4} V''(z) + \frac{r^4}{64} \frac{\partial^4 V}{\partial z^4} + \dots$$

$$\Rightarrow E_r(r, z) = \frac{r}{2} V''(z) - \frac{r^3}{16} \frac{\partial^4 V(z)}{\partial z^4} + \dots$$

$$\nabla E_z(r, z) = -V'(z) + \frac{r^2}{4} V'''(z) + \dots$$

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RETURNING TO THE RADIAL COMPONENT OF THE
MOMENTUM EQUATION IN CYLINDRICAL COORDINATES (EQ I):

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = q(E_r + r \dot{\theta} B_z - \dot{z} B_\theta) \quad (I)$$

for the external field use

$$E_{r\text{-ext}} = \frac{r}{z} V'' + O(r^3)$$

$$B_{z\text{-ext}} = B_z(z) + O(r^3)$$

$$B_{\theta\text{-ext}} = 0 \quad \left[\text{since } \frac{\partial \Phi_{\text{ext}}}{\partial \theta} = 0 \right]$$

WE LET:

$$\underline{B} = \underline{B}_{\text{ext}} + \underline{B}_{\text{self}}$$

$$\underline{E} = \underline{E}_{\text{ext}} + \underline{E}_{\text{self}}$$

PARAXIAL RAY EQUATION

$$(I) \Rightarrow \frac{d}{dt} (\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = q \left(\frac{V''}{z} r + r \dot{\theta} B \right) + q (E_r^{\text{self}} - v_z B_\theta^{\text{self}})$$

↑
INERTIAL

↑
CENTRIFUGAL

↑
 E_r
external

↑
 $v_z B_\theta$
external

↑ ↑
SELF
FIELDS

Now use s as independent variable $v_z dt = ds$

$$v_z \frac{d}{ds} (\gamma m v_z r') - \gamma m v_z^2 r \theta'^2 = q \left(\frac{V''}{z} r + r v_z \theta' B \right) + q (E_r^{\text{self}} - v_z B_\theta^{\text{self}})$$

EXPANDING 1st term and $v_z \cong v$; AND DIVIDING BY $\gamma m v^2$:

$$r'' - r \theta'^2 + \frac{\gamma'}{\beta \gamma} r' = \frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{z} r + r \beta c \theta' B + E_r^{\text{self}} - v_z B_\theta^{\text{self}} \right) \quad (P1)$$

Using CANONICAL MOMENTUM, eliminate θ' via

$$\theta' = \frac{p_\theta - \frac{q\psi}{2\pi}}{\gamma m r^2 \beta c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{qB}{2\gamma \beta m c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma \beta c}$$

where we define $\omega_c \equiv \frac{qB}{m}$

ADDING THE TWO θ' TERMS IN THE EQUATION (P1)

$$\begin{aligned} -r \theta'^2 - \frac{r \omega_c \theta'}{\gamma \beta c} &= \frac{-p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} - \frac{r \omega_c^2}{4 \gamma^2 \beta^2 c^2} \\ &\quad - \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} + \frac{r \omega_c^2}{2 \gamma^2 \beta^2 c^2} \\ &= \frac{-p_\theta^2}{\gamma^2 m^2 r^3 \beta^2 c^2} + \frac{r \omega_c^2}{2 \gamma^2 \beta^2 c^2} \end{aligned}$$

So equation (P1) becomes:

$$r'' + \frac{\gamma'}{\beta^2 \gamma} r' = \frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{z} r \right) + \frac{r \omega_c^2}{2 \gamma^2 \beta^2 c^2} + \frac{p_0^z}{\gamma^2 m^2 \beta^2 c^2} + \frac{q}{\gamma m \beta^2 c^2} \left[E_r^{\text{self}} - v_z B_\theta^{\text{self}} \right] \quad (P2)$$

Now $\gamma' m c^2 = q \frac{\mathbf{E} \cdot \mathbf{v}}{v_z} \Rightarrow \gamma' = \left(V'' + \frac{\partial^2 \phi^{\text{self}}}{\partial z^2} \right) \frac{q}{m c^2}$

CALCULATING $\frac{q}{\gamma m \beta^2 c^2} \left[\frac{V''}{z} r + E_r^{\text{self}} - v_z B_\theta^{\text{self}} \right]$:

$$\nabla^2 \phi^{\text{self}} = -\frac{\rho}{\epsilon_0} \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) = -\frac{\rho}{\epsilon_0} - \frac{\partial^2 \phi^{\text{self}}}{\partial z^2}$$

$$\Rightarrow \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) = -\frac{r \rho(r)}{\epsilon_0} - \frac{r \partial^2 \phi^{\text{self}}}{\partial z^2}$$

$$r \frac{\partial \phi}{\partial r} = -\frac{1}{2\pi \epsilon_0} \int_0^r 2\pi r' \rho(r') dr - \frac{r^2}{2} \frac{\partial^2 \phi^{\text{self}}}{\partial z^2}$$

$$= -\frac{1}{2\pi \epsilon_0} \lambda(r) - \frac{r^2}{2} \frac{\partial^2 \phi^{\text{self}}}{\partial z^2}$$

$$\Rightarrow E_r^{\text{self}} = \frac{\lambda(r)}{2\pi \epsilon_0 r} + \frac{r}{2} \frac{\partial^2 \phi^{\text{self}}}{\partial z^2}$$

$$\nabla \times \underline{B} = \mu_0 \underline{J} \Rightarrow 2\pi r B_\theta = \mu_0 \int_0^r 2\pi r' J_z(r') dr = \mu_0 v_z \lambda(r)$$

$$B_\theta^{\text{self}} = \frac{\mu_0 v_z \lambda(r)}{2\pi r} = \frac{v_z}{c^2} \frac{\lambda(r)}{2\pi \epsilon_0 r}$$

$$\left[\frac{V''}{z} r + E_r^{\text{self}} - v_z B_\theta^{\text{self}} \right] = \left[\frac{r}{z} \left(V'' + \frac{\partial^2 \phi^{\text{self}}}{\partial z^2} \right) + \left(1 - \frac{v_z^2}{c^2} \right) \frac{\lambda(r)}{2\pi \epsilon_0 r} \right]$$

$\underbrace{\hspace{10em}}_{-\frac{m c^2}{q} \gamma''} \quad \underbrace{\hspace{10em}}_{1/\gamma^2}$

So equation (P2) becomes: "THE PARAXIAL RAY EQUATION:"

$$r'' + \frac{\gamma'}{\beta^2 \gamma} r' + \frac{\gamma''}{2\beta^2 \gamma} r + \left(\frac{\omega c}{2\gamma \beta c} \right)^2 r - \left(\frac{p_0}{\gamma \beta m c} \right)^2 \frac{1}{r^3} - \frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi \epsilon_0 r} = 0$$

INERTIAL
E_r
(CONVERGENCE
OF
FIELD
LINES)
V_b B_z
- CENTRIFUGAL
CENTRIFUGAL
SELF
FIELD

MOMENT EQUATIONS

Vlasov eqn: $\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} = 0$

Let $g = g(x, x', y, y')$; $N = \iiint f dx dx' dy dy'$

MULTIPLY Vlasov equation by g & $\frac{1}{N} \iiint dx dx' dy dy'$

$\int dx dx' dy dy' \left[g \frac{\partial f}{\partial s} + g x' \frac{\partial f}{\partial x} + g x'' \frac{\partial f}{\partial x'} + g y' \frac{\partial f}{\partial y} + g y'' \frac{\partial f}{\partial y'} \right] = 0$

$\Rightarrow \frac{d}{ds} \langle g \rangle + \underbrace{\frac{1}{N} \iiint g f \left[- \frac{\partial g}{\partial x} f x' + \dots \right]}_{\text{INTEGRATE BY PARTS}} = 0$

$\rightarrow 0 = \langle x' \frac{\partial g}{\partial x} \rangle$

$\Rightarrow \frac{d}{ds} \langle g \rangle = \langle x' \frac{\partial g}{\partial x} \rangle + \langle x'' \frac{\partial g}{\partial x'} \rangle + \langle y' \frac{\partial g}{\partial y} \rangle + \langle y'' \frac{\partial g}{\partial y'} \rangle$

But $\frac{dg}{ds} = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial x'} x'' + \frac{\partial g}{\partial y} y' + \frac{\partial g}{\partial y'} y''$

$\Rightarrow \frac{d}{ds} \langle g \rangle = \langle g' \rangle$

So $\frac{d}{ds} \langle x^2 \rangle = 2 \langle x x' \rangle$

$\frac{d}{ds} \langle x'^2 \rangle = 2 \langle x' x'' \rangle$ etc...

$\frac{d}{ds} \langle x x' \rangle = \langle x x'' \rangle + \langle x'^2 \rangle$

ENVELOPE EQUATION FOR AXISYMMETRIC BEAMS

$$\text{LET } r_b^2 = 2 \langle r^2 \rangle = 2(\langle x^2 \rangle + \langle y^2 \rangle) = 4 \langle x^2 \rangle$$

for an
axisymmetric
beam

$$2r_b r_b' = 4 \langle r r' \rangle \quad \Rightarrow \quad r_b' = \frac{2 \langle r r' \rangle}{r_b}$$

$$r_b'' = \frac{2 \langle r r'' \rangle + 2 \langle r'^2 \rangle}{r_b} - \frac{2 \langle r r' \rangle}{r_b^2} \left(\frac{2 \langle r r' \rangle}{r_b} \right)$$

$$= 2 \frac{\langle r r'' \rangle}{r_b} + \frac{4 \langle r'^2 \rangle}{r_b} - 4 \frac{\langle r r' \rangle^2}{r_b^2}$$

WHAT IS $\langle r r'' \rangle$?

RECALL EQUATION P1 (ON PATH TO MAXWELL RAY EQUATION):

$$n'' - n\theta'^2 + \frac{\gamma'}{\beta^2 \gamma} n' = \frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{2} r + r \rho c \theta' B + E_r^{self} - \frac{1}{2} \beta^2 \dot{B}^2 \right)$$

P1 may be rewritten:

$$n'' - n\theta'^2 + \frac{\gamma'}{\beta^2 \gamma} n' = \frac{q}{\gamma m \beta^2 c^2} \left[\frac{-m c^2}{q} \gamma'' \frac{r}{2} + \frac{\lambda(r)}{\gamma^2 2 \pi \epsilon_0 r} + r \rho c \theta' B \right]$$

$$n'' + \frac{\gamma'}{\beta^2 \gamma} n' + \frac{\gamma''}{2 \beta^2 \gamma} r - \frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2 \pi \epsilon_0 r} - \frac{\omega_c}{\gamma \rho c} \theta' r - n\theta'^2 = 0$$

What is $\langle r n'' \rangle$?

$$\langle r n'' \rangle + \frac{-\omega_c}{\gamma \rho c} \langle \theta' r^2 \rangle - \langle n^2 \theta'^2 \rangle + \dots = 0$$

$$\langle r n'' \rangle = \gamma^2 m^2 \beta^2 c^2 \langle n^2 \theta'^2 \rangle + \frac{\omega_c^2}{4} m^2 \langle r^2 \rangle + \omega_c \gamma m^2 \beta c \langle r n \theta' \rangle$$

$$\Rightarrow \frac{-\omega_c}{\gamma \rho c} \langle \theta' r^2 \rangle = \frac{-\omega_c}{\gamma \rho c} \left[\frac{\langle r n \rangle^2}{\omega_c \gamma m^2 \beta c \langle r^2 \rangle} - \frac{\omega_c \langle r^2 \rangle}{4 \gamma m \rho c} - \frac{\gamma \rho c \langle n^2 \theta'^2 \rangle}{\omega_c \langle r^2 \rangle} \right]$$

$$\Rightarrow \langle r n'' \rangle = \frac{\langle r n \rangle^2}{\gamma^2 m^2 \beta^2 c^2 \langle r^2 \rangle} - \frac{\omega_c^2 \langle r^2 \rangle}{4 \gamma m \rho c} - \frac{\langle n^2 \theta'^2 \rangle}{\langle r^2 \rangle} + \langle n^2 \theta'^2 \rangle + \dots = 0$$

⇒

$$\langle r r'' \rangle = \frac{\gamma'}{\beta^2 \gamma} \langle r r' \rangle + \frac{\gamma''}{2\beta^2 \gamma} \langle r^2 \rangle - \frac{q}{\gamma^3 m v_E^2} \frac{\langle \lambda(r) \rangle}{2\pi \epsilon_0} + \frac{\langle p_0 \rangle^2}{(\gamma m \beta c)^2 \langle r^2 \rangle} - \frac{\omega_c^2 \langle r^2 \rangle}{4(\gamma^2 \beta c)^2} - \frac{\langle r^2 \theta'^2 \rangle}{\langle r^2 \rangle} + \langle r^2 \theta''^2 \rangle$$

$$r_b'' = \frac{2 \langle r r'' \rangle}{r_b} + \frac{4 \langle r^2 \rangle \langle r''^2 \rangle - 4 \langle r r' \rangle^2}{r_b^3}$$

$$\begin{aligned} &= \frac{\gamma'}{\beta^2 \gamma} \frac{2 \langle r r' \rangle}{r_b} + \frac{\gamma''}{2\beta^2 \gamma} \frac{2 \langle r^2 \rangle}{r_b} - \frac{2q}{\gamma^3 m v_E^2} \frac{\langle \lambda(r) \rangle}{2\pi \epsilon_0} \frac{1}{r_b} \\ &+ \frac{\langle p_0 \rangle^2}{(\gamma m \beta c)^2} \frac{2}{\langle r^2 \rangle r_b} - \frac{\omega_c^2}{4(\gamma \beta c)^2} \frac{2 \langle r^2 \rangle}{r_b} - \frac{2 \langle r^2 \theta'^2 \rangle}{r_b \langle r^2 \rangle} \\ &+ \frac{2 \langle r^2 \theta''^2 \rangle}{r_b} + \frac{4 \langle r^2 \rangle \langle r''^2 \rangle - 4 \langle r r' \rangle^2}{r_b^3} \end{aligned}$$

Using $r_b^2 \equiv 2 \langle r^2 \rangle$ & $r_b' = \frac{2 \langle r r' \rangle}{r_b}$

ENVELOPE EQUATION

$$\Rightarrow \left[r_b'' + \frac{\gamma'}{\beta^2 \gamma} r_b' + \frac{\gamma''}{2\beta^2 \gamma} r_b + \left(\frac{\omega_c}{2\gamma \beta c} \right)^2 r_b + \frac{-4 \langle p_0 \rangle^2}{(\gamma m \beta c)^2 r_b^3} - \frac{E_r^2}{r_b^3} - \frac{Q}{r_b} = 0 \right]$$

WHERE $E_r^2 = 4(\langle r^2 \rangle \langle r''^2 \rangle - \langle r r' \rangle^2) + \langle r^2 \rangle \langle r^2 \theta''^2 \rangle - \langle r^2 \theta'^2 \rangle^2$

ENVELOPE EQUATION -- CONTINUED

$$r_b'' + \frac{\gamma'}{\beta^2 \gamma} r_b' + \frac{\gamma''}{2\beta^2 \gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c} \right)^2 r_b - \frac{4\langle p_0 \rangle^2}{(\gamma m \beta c)^2} r_b^3 - \frac{E_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

COMPARE WITH THE SINGLE PARTICLE PARAXIAL RAY EQUATION:

$$r'' + \underbrace{\frac{\gamma'}{\beta \gamma}}_{\text{INERTIAL}} r' + \underbrace{\frac{\gamma''}{2\beta \gamma}}_{E_r} r + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c} \right)^2}_{V_0 B_z - \text{CENTRIFUGAL}} r - \underbrace{\left(\frac{p_0}{\gamma m \beta c} \right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\chi(r)}{2\pi \epsilon_0 r}}_{E_r - V_0 B_z \text{ self field}} = 0$$

$$E_r^2 = 4(\langle r^2 \rangle \langle r'^2 \rangle - \langle r r' \rangle^2 + \langle r^2 \rangle \langle r^2 \theta'^2 \rangle - \langle r^2 \theta' \rangle^2)$$

NOTE THAT FOR AXISYMMETRIC BEAMS ($\rho = \rho(r)$ ONLY)

$$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle = 2\langle x^2 \rangle$$

$$\Rightarrow 2\langle r r' \rangle = 4\langle x x' \rangle$$

$$\& \langle x'^2 \rangle + \langle y'^2 \rangle = 2\langle x'^2 \rangle = \langle r'^2 \rangle + \langle r^2 \theta'^2 \rangle$$

DEFINE $E_x^2 = 16(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2)$

$$\Rightarrow \boxed{E_r^2 = E_x^2 - 4\langle r^2 \theta' \rangle^2}$$

EXAMPLES OF
SYSTEMS WITH AXIAL SYMMETRY

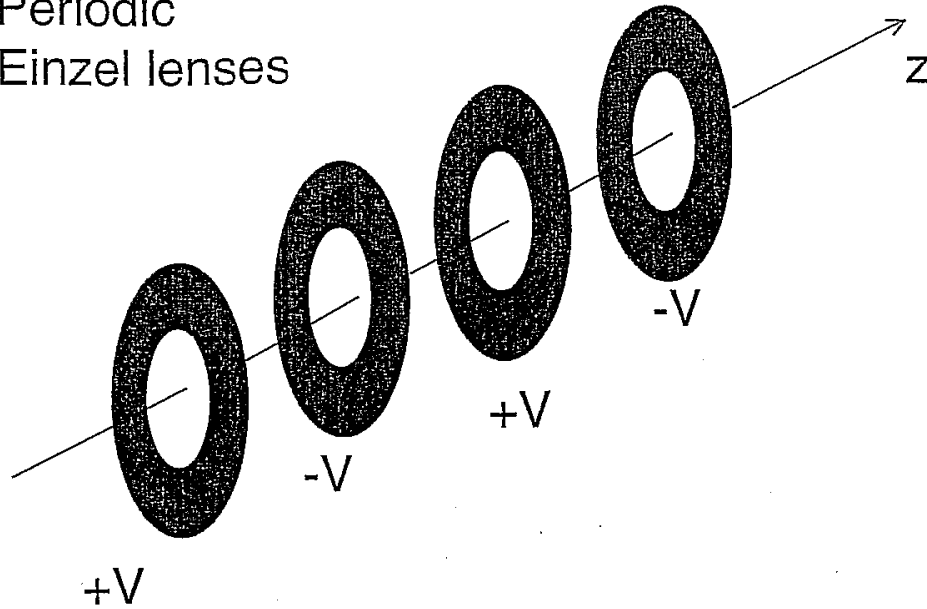
- PERIODIC SOLENOIDS
- EINZEL LENSES
- CONTINUOUS FOCUSING

EXAMPLES OF
SYSTEMS WITHOUT AXIAL SYMMETRY

- ELECTRIC OR MAGNETIC QUADRUPOLE
- ⇒ USE CARTESIAN COORDINATES WITH
ELLIPTICAL SPACE CHARGE SYMMETRY

EXAMPLES OF AXISYMMETRIC SYSTEMS

Periodic Einzel lenses



PERIODIC SOLENOIDS

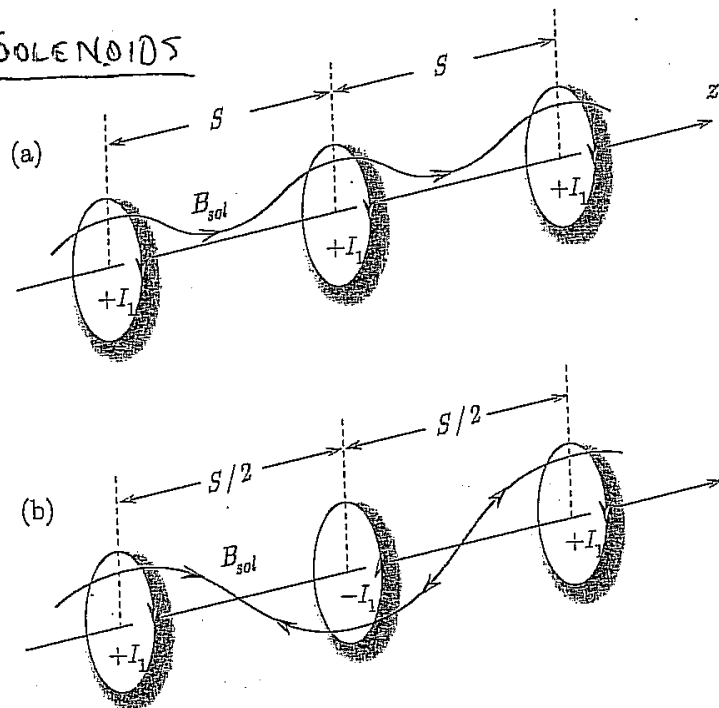


Figure 3.2. Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length S . In Fig. 3.2 (a), successive coils are spaced by S and have the same current polarity $+I_1, +I_1, \dots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \dots$.

(FIGURE FROM DAVIDSON & QIN 2003) P. 55 "PHYSICS OF INTENSE CHARGED PARTICLE BEAMS IN HIGH ENERGY ACCELERATORS"

EXAMPLE OF NON-AXISYMMETRIC SYSTEM

Figure from
Davidson & Qin, 2003.

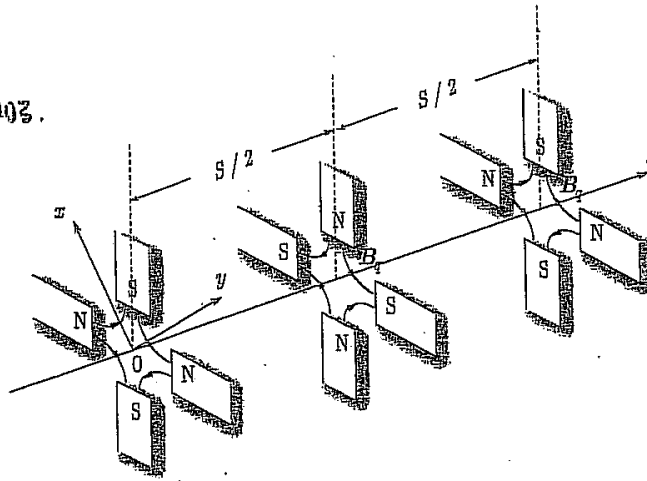


Figure 3.1. Schematic of magnet sets producing an alternating-gradient quadrupole field with axial periodicity length S .

NON-AXISYMMETRIC SYSTEM

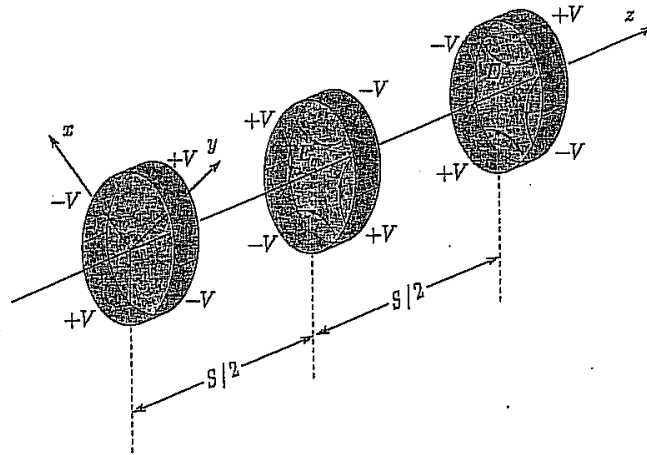


Figure 3.3. Schematic of conductor configuration with applied voltages producing an alternating-gradient quadrupole electric field with axial periodicity length S .

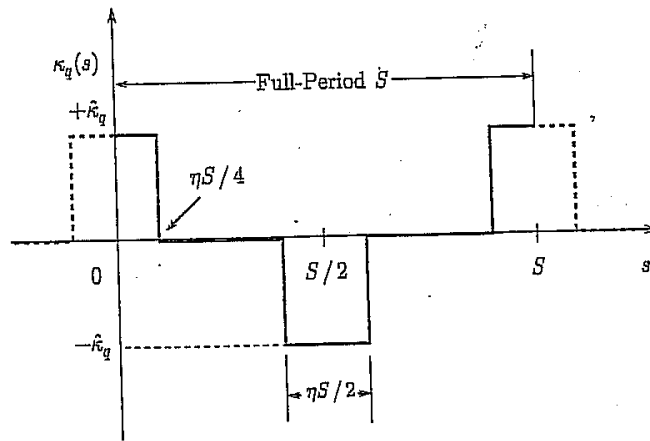


Figure 3.7. Alternating step-function model of a periodic quadrupole lattice with filling factor η for the lens elements. The figure shows a plot of the quadrupole coupling coefficient $\kappa_q(s)$ versus s for one full period (S) of the lattice. Such a configuration is often called a FODO transport lattice (acronym for focusing-off-defocusing-off).

FIGURES FROM DAVIDSON & QIN, 2003

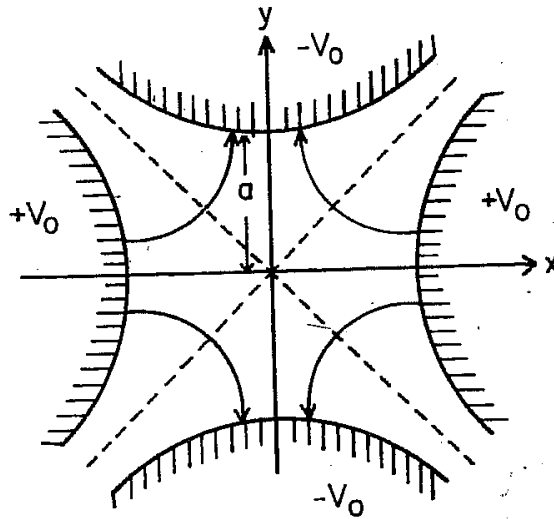
J. BANWANI
 (22)

2 ≡ BEAM OPTICS AND FOCUSING SYSTEMS WITHOUT SPACE CH

FROM
 REISER, p. 112

$$E_x = -E'x$$

$$E_y = E'y$$



$$F_x = -qE'x$$

$$F_y = qE'y$$

ELECTROSTATIC
 QUADS

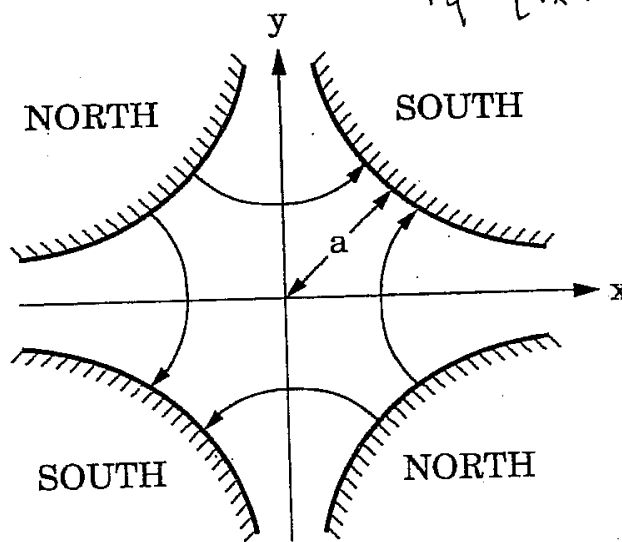
Figure 3.15. Electrodes and force lines in an electrostatic quadrupole.

$$B_x = B'y$$

$$B_y = B'x$$

$$F_x = -qv_z B'x$$

$$F_y = qv_x B'y$$



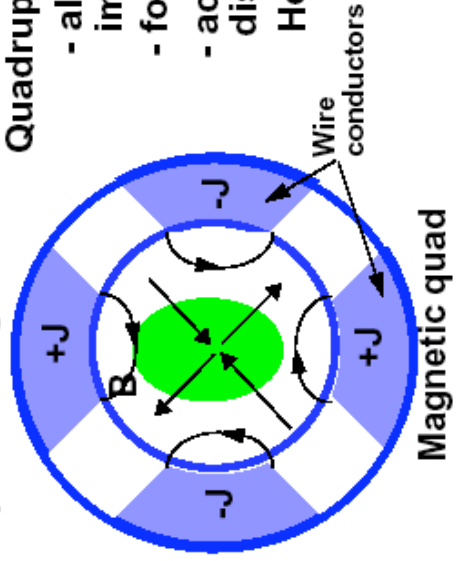
MAGNETIC
 QUADS

Heavy ion accelerators use alternating gradient quadrupoles to confine the beams

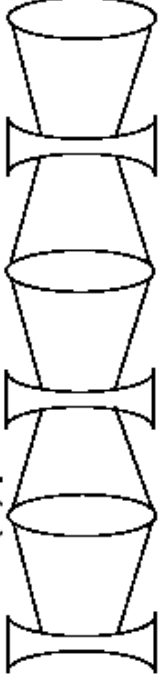
Space-charge forces and thermal forces act to expand beam

Quadrupoles (magnetic or electric):

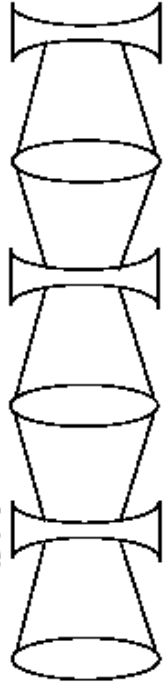
- alternately provide inward then outward impulse
- focus in one plane and defocus in other
- act as linear lenses. (Force proportional to distance from axis).



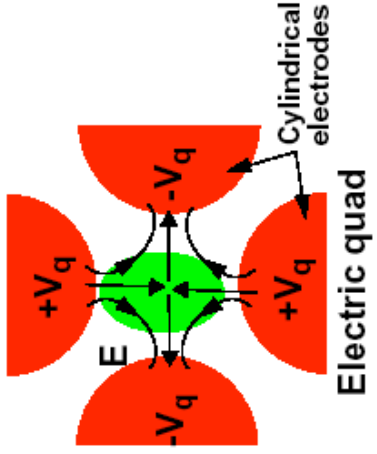
Horizontal (x) plane:



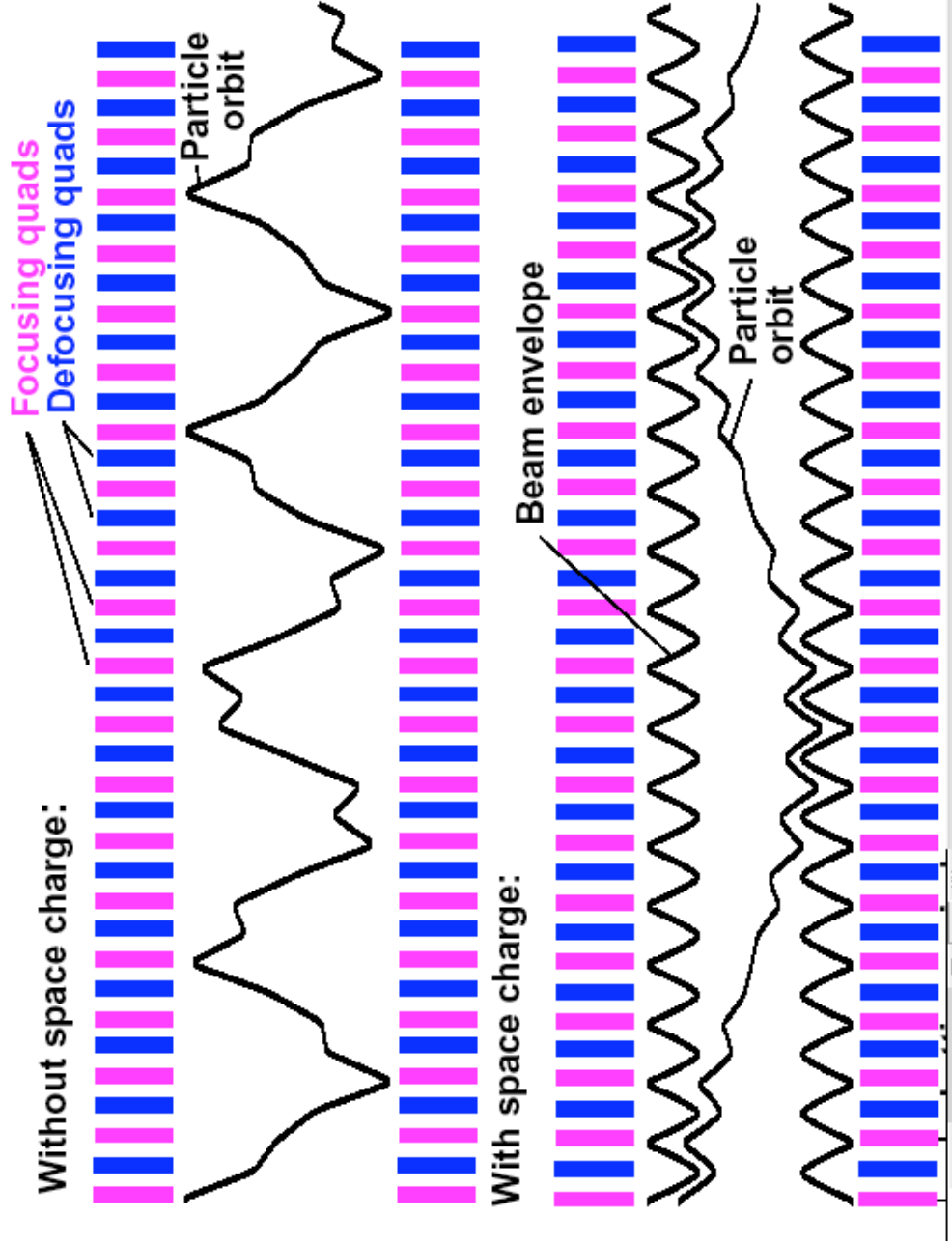
Vertical (y) plane:



Average displacement is larger in focusing lenses so the net effect is focusing.



Space charge reduces betatron phase advance



ENVELOPE EQUATIONS FOR NON-AXISYMMETRIC SYSTEMS

(25)

$$r_x^2 \equiv 4 \langle x^2 \rangle \quad r_y^2 \equiv 4 \langle y^2 \rangle$$

$$2 r_x r_x' = 8 \langle x x' \rangle$$

$$r_x' = \frac{4 \langle x x' \rangle}{r_x}$$

$$\begin{aligned} r_x'' &= \frac{4 \langle x x'' \rangle}{r_x} + \frac{4 \langle x'^2 \rangle}{r_x} - \frac{4 \langle x x' \rangle}{r_x^2} r_x' \\ &= \frac{4 \langle x x'' \rangle}{r_x} + \frac{16 \langle x'^2 \rangle \langle x^0 \rangle}{r_x^2} - \frac{16 \langle x x' \rangle^2}{r_x^3} \end{aligned}$$

DEFINE $E_x^2 = 16 (\langle x'^2 \rangle \langle x^0 \rangle - \langle x x' \rangle^2)$

$$\Rightarrow \boxed{r_x'' = \frac{4 \langle x x'' \rangle}{r_x} + \frac{E_x^2}{r_x^3}}$$

SO HOW DO WE CALCULATE $\langle x x'' \rangle$?

RETURN TO SINGLE PARTICLE EQUATION (IN CARTESIAN COORDINATES)

$$\frac{d}{dt} (\gamma m \dot{x}) = \gamma m \ddot{x} = q (E_x + \dot{y} B_z - \dot{z} B_y)$$

\downarrow
 x''
± similarly
 y''

\downarrow
QUADRUPOLE FOCUSING
SPACE-CHARGE OF ELLIPTICAL
BEAMS

TO BE CONTINUED ...

J. BARWAD

QUADRUPOLE FOCUSING

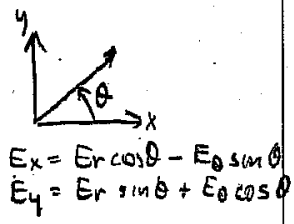
Now, relax radial symmetry:

For $\nabla \cdot B = 0$ or $\nabla \times B = 0$

EXPAND FIELD IN CYLINDRICAL "MULTIPOLES":

$$E_r, B_r = \sum_{n=1}^{\infty} f_n r^{n-1} \cos(n\theta)$$

$$E_\theta, B_\theta = \sum_{n=1}^{\infty} f_n r^{n-1} \sin(n\theta)$$



$n=1 \Rightarrow$ dipole $\begin{cases} E_r = f_1 \cos \theta \\ E_\theta = -f_1 \sin \theta \end{cases} \Rightarrow \begin{cases} E_x = f_1 \\ E_y = 0 \end{cases}$

$n=2 \Rightarrow$ quadrupole $\begin{cases} E_r = f_2 r \cos 2\theta \\ E_\theta = -f_2 r \sin 2\theta \end{cases} \Rightarrow \begin{cases} E_x = f_2 x \\ E_y = -f_2 y \end{cases}$

NOTE: ABOVE EXPANSION IS VALID WHEN E or $B \neq$ function(z).
 FOR MAGNETS OF FINITE AXIAL EXTENT, FOR EACH FUNDAMENTAL n -pole, A SET OF HIGHER ORDER MULTIPOLES WITH SAME AZIMUTHAL SYMMETRY ARE REQUIRED TO SATISFY $\nabla^2 \phi = 0$.

FOR EXAMPLE FOR A FUNDAMENTAL QUADRUPOLE THE FIELD MAY BE EXAMINED:

$$E_r = \sum_{\nu=0}^{\infty} f_{2,\nu}(z) [1+\nu] r^{1+2\nu} \cos[2\theta]$$

$$E_\theta = \sum_{\nu=0}^{\infty} -f_{2,\nu}(z) r^{1+2\nu} \sin[2\theta]$$

$$E_z = \sum_{\nu=0}^{\infty} \frac{1}{2} \frac{df_{2,\nu}}{dz} r^{2+2\nu} \cos 2\theta$$

with $f_{2,\nu+1}(z) = \frac{-1}{4(\nu+1)(\nu+3)} \frac{d^2 f_{2,\nu}(z)}{dz^2}$

SEE LUND, S. M. (1996)
 FOR EXAMPLE. HIF note 96-1
 LLNL.

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