USPAS Accelerator Physics
Problem Set 3 - 100 pts.

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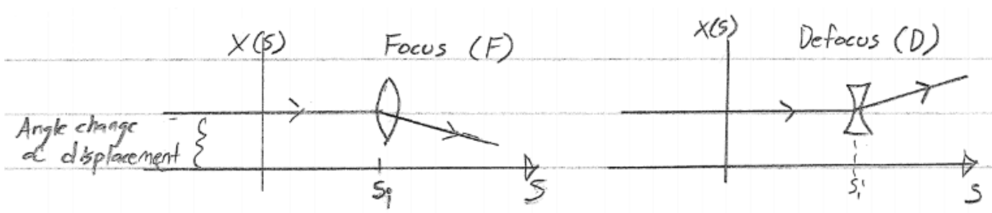
Problem 1

P010 Thin Lens FODO Phase Advance and Stability 20 pts.

A thin lens changes the angle of a particle trajectory but not the coordinate:



This action can be specified by transfer matrices applied at $s = s_i$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathbf{M}(s|s_i) \begin{bmatrix} x_i \\ x'_i \end{bmatrix}$$

Focusing:

$$\mathbf{M}_F = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \qquad f > 0$$

Defocusing:

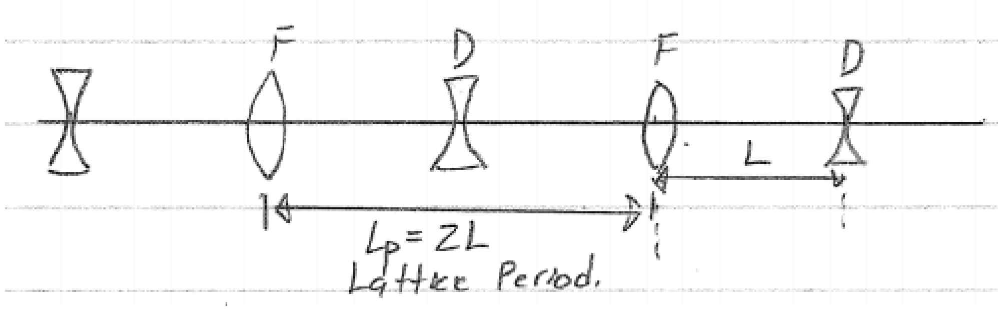
$$\mathbf{M}_D = \begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix} \qquad f > 0$$

A free space drift of length L has a transport matrix:

$$\mathbf{M}_0 = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

Consider a lattice of period $2L$ made up of equally spaced F and D lenses with equal values of f.

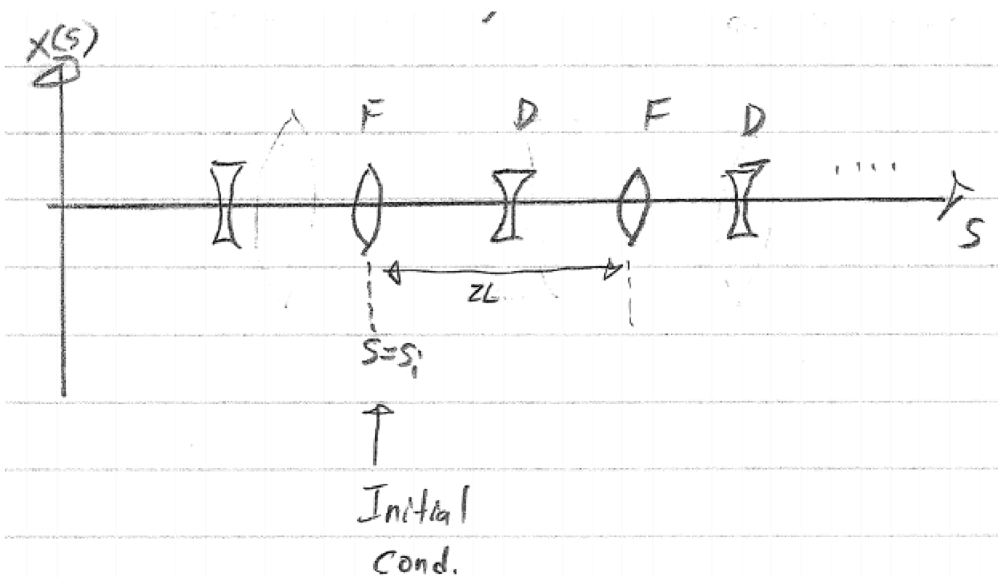
a) 10 pts: Calculate $\cos\sigma_0$ where σ_0 is the particle phase advance. Express the answer in terms of L/f. Also, determine the range of f for which the particle orbit is stable.



b) 10 pts: For the case of f chosen to correspond to the stability limit, sketch the motion of a particle with initial conditions:

$$\lim_{s \to s_i} x(s) = x_0$$
$$\lim_{s \to s_i} x'(s) = x_o/L$$

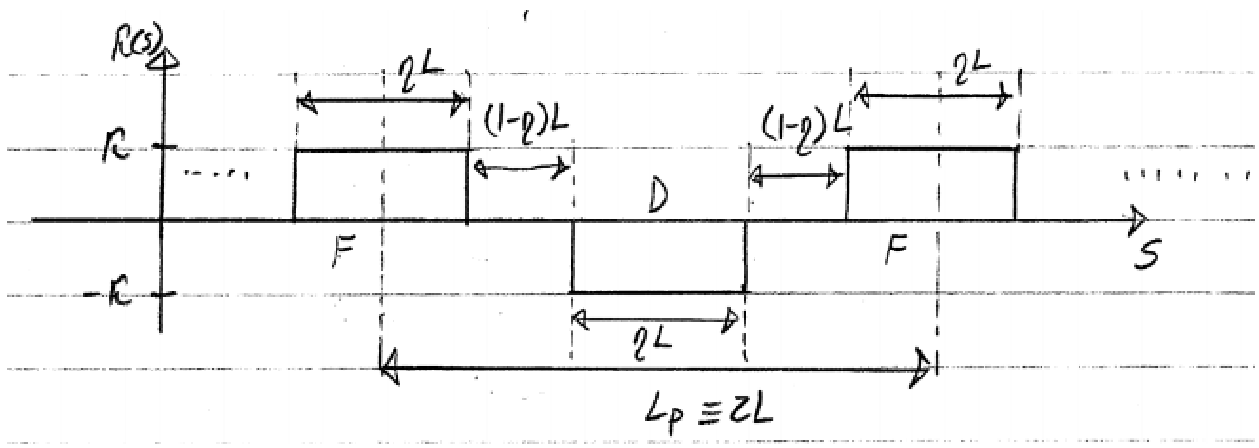
where $s = s_i$ is the axial location of a focusing thin lens kick, and $s \to s_i^-$ is just before the kick. Sketch the particle orbit for focusing strength slightly off from the stability limit into the undstable region. Superimpose the orbit sketch on a diagram of the lattice (see below):



Problem 2

P007 Phase Advance 35 pts.

Consider a periodic FODO lattice:



$L_p = 2L =$ Lattice period
$\eta L =$ Quadrupole length
$(1-\eta)L =$ Drift length
$\eta =$ Quadrupole occupancy $, 0 < \eta \leq 1$
$\kappa =$ Quadrupole strength

a) 5 pts: Write the transfer matrices $\mathbf{M}(s|s_i)$ for each section of the periodic lattice in terms of $\theta \equiv \sqrt{\kappa}\eta L$. Use the results from class.

$\mathbf{M_f}$: Transfer through focusing quadrupole
$\mathbf{M_O}$: Transfer through drift
$\mathbf{M_D}$: Transfer through defocusing quadrupole
$\mathbf{M_O}$: Transfer through drift

b) 5 pts: Write the transfer matrix, $\mathbf{M}(s_i + L_p|s_i)$, through one lattice period starting from the left side of a focus quadrupole. No need to fully expand.

c) 10 pts: Show that the phase advance σ_0 of a particle through this lattice period

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Trace}\left[\mathbf{M}(s_i + L_p|s_i)\right]$$

can be expressed as:

$$\cos\sigma_0 = \cos\theta\cosh\theta + \frac{1-\eta}{\eta}\theta\left[\cos\theta\sinh\theta - \sin\theta\cosh\theta\right] - \frac{1}{2}\frac{(1-\eta)^2}{\eta^2}\theta^2\sin\theta\sinh\theta$$

Hint: Only calculate the elements of $\mathbf{M}$ that you need.

d) 3 pts: Will it matter where the lattice period is started in the calculation of σ_0 in part c)? Why?

e) 5 pts: For $\theta << 1$ (thin lens limit), show that:

$$\cos\sigma_0 \simeq 1 - \frac{1}{2}\left(1 - \frac{2}{3}\eta\right)\frac{\theta^4}{\eta^2}$$

f) 5 pts: If $\sigma_0 << 1$ and $\eta << 1$ show that:

$$\sigma_0 \simeq \eta |\kappa| L^2$$

g) 2 pts: If one wanted to model a FODO focusing lattice by a continuous focusing channel with $\kappa(s) = k_{\beta_0}^2 =$ constant, how could one choose $k_{\beta_0}^2$ based on part f)?

Problem 3

P011 Courant-Snyder Ellipse 30 pts.

In class we derived the single-particle Courant-Snyder invariant:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon = \text{const.}$$

where:

$$\begin{aligned}
\beta(s) &= w^2(s) \\
\alpha(s) &= -w(s)w'(s) \\
\gamma(s) &= \frac{1}{w^2(s)} + w'(s)^2 = \frac{1+\alpha^2(s)}{\beta(s)}
\end{aligned}$$

Derive the critical values of the ellipse indicated on the figure below:

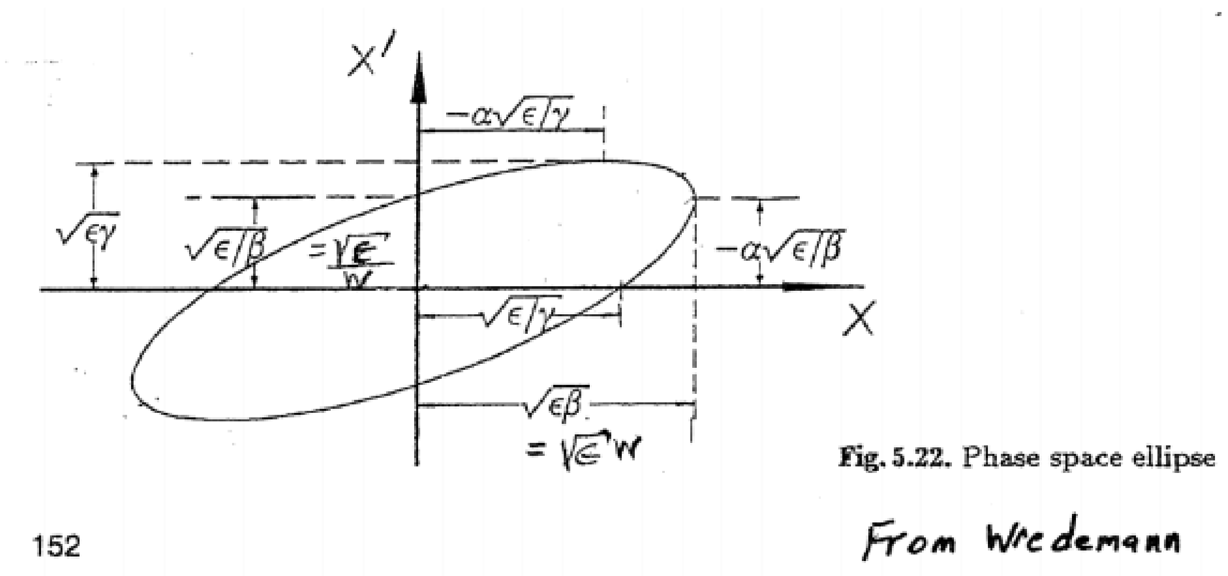


Fig. 5.22. Phase space ellipse

From Wiedemann

Hint: To avoid messy algebra, take a differential of the constraint equation

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \text{const.}$$

as

$$\Rightarrow \quad 2\gamma x dx + 2\alpha x dx' + 2\alpha x' dx + 2\beta x' dx' = 0$$

and use this result to find turning points.

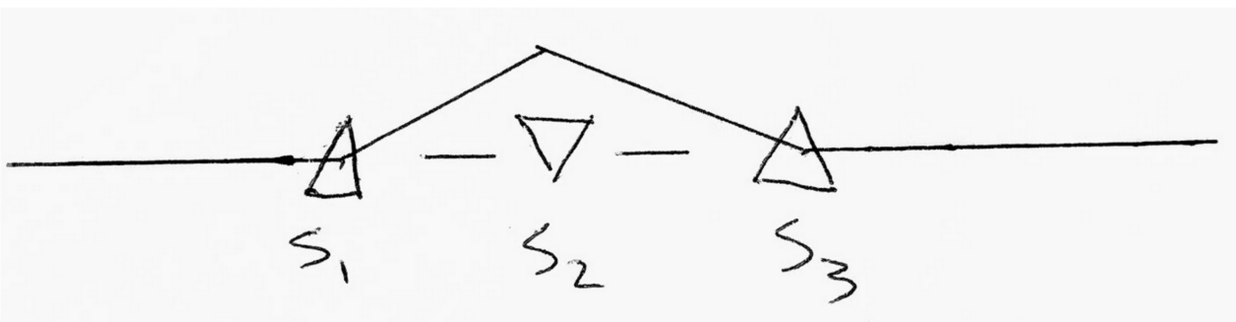
Problem 4

P035 3-Bump 15 pts.

The "3-bump" method is commonly used in ring accelerators for local orbit correction over one lap. Consider three axially short (kick) corrector dipoles in a storage ring, each provide kicking angle θ_i ($i = 1, 2, 3$). The dipoles are located at axial coordinates $s = s_i$ ($s_1 < s_2 < s_3$), where the betatron functions are β_i and the phase advances are ψ_i ($\psi_1 < \psi_2 < \psi_3$). The phase-advance is measured from the reference orbit at $s = s_0$. The "three-bump" uses the three corrector magnets to create a non-zero closed orbit between the correctors ($s_1 < s < s_3$) while maintaining a zero orbit outside the region.

Prove that the necessary angles θ_2 and θ_3 are:

$$\theta_2 = -\frac{\sqrt{\beta_1}\theta_1}{\sqrt{\beta_2}\sin(\psi_{32})}\sin\psi_{31}$$
$$\theta_3 = \frac{\sqrt{\beta_1}\theta_1}{\sqrt{\beta_3}\sin(\psi_{32})}\sin\psi_{21}$$



Hint: Use the Green function of dipole error in the perturbed linear motion session. Make sure that the combination of the effect of the three corrector has zero effect outside the region between s_1 and s_3. The phase advance of a particle over a lap of the ring is $\psi = 2\pi\nu$, where ν is the tune.