Solenoid Focusing

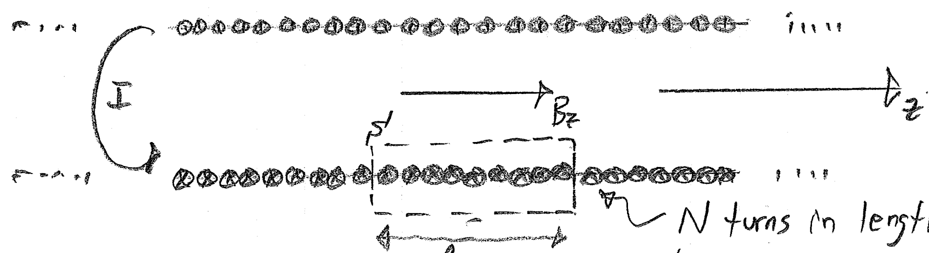
Solenoid magnets commonly employed to focus low energy beams and are distinct from magnetic quadrupole optics.

- FRIB uses superconducting solenoids for beam focusing in all 3 linac segments. Max $B_z \sim 8$ to 9 Tesla.

Fields

Consider an iron-free, infinitely long solenoid made up of current loops in vacuum:

Thin Coil Solenoid



Reminder: E&M analysis
$B_z = \begin{cases} \text{const} & \text{inside} \\ 0 & \text{outside} \end{cases}$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$
$$\nabla \cdot \vec{B} = 0$$

$$\Rightarrow \int_{S'} \nabla \times \vec{B} \cdot \hat{\theta}\, d^2x = \mu_0 \int_{S'} \vec{J} \cdot \hat{\theta}\, d^2x$$

$$\int_{\partial S'} \vec{B} \cdot d\vec{\ell} = \mu_0 N \cdot I$$

$$B_z \ell = \mu_0 N I$$

$\frac{N}{\ell}$ = # turns per unit axial length

$$\boxed{B_z = \mu_0 \frac{N}{\ell} I}$$ in bore.

But orbit of a particle in a solenoid is helical!

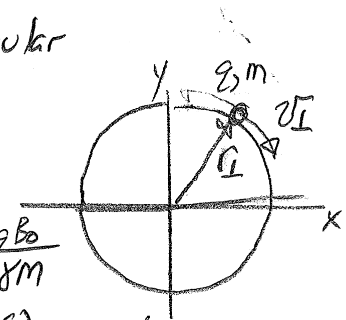
Perp ($\perp$): Let $v_\perp$ = velocity $\perp$ to $B_z = B_0$ = const, orbit in $\perp$ plane is circular

$$\gamma m \frac{v_\perp^2}{r_\perp} = q|v_\perp| B_0 = |q\vec{v}\times\vec{B}|_\perp$$

$v_\perp$ = const

$$r_\perp = \frac{\gamma m |v_\perp|}{q B_0} = \frac{\gamma |v_\perp|}{\omega_c} = \text{const}$$

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2\pi r_\perp / |v_\perp|} = \frac{|v_\perp|}{r_\perp} = \frac{q B_0}{\gamma m} = \frac{\omega_c}{\gamma} = \text{const}$$



$$x(t) = r_\perp \cos \omega t$$
$$y(t) = r_\perp \sin \omega t$$

choice $t=0$ made on initial conditions

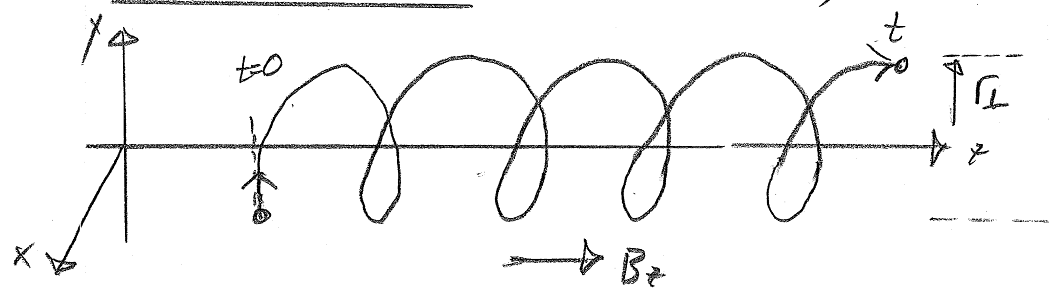
$\omega_c = \frac{qB}{m}$ = cyclotron freq.

Parallel (z): Let v_z = velocity $\parallel$ to $B_z = B_0$, free streaming orbit

$$z(t) = z_0 + v_z t$$

~ free streaming

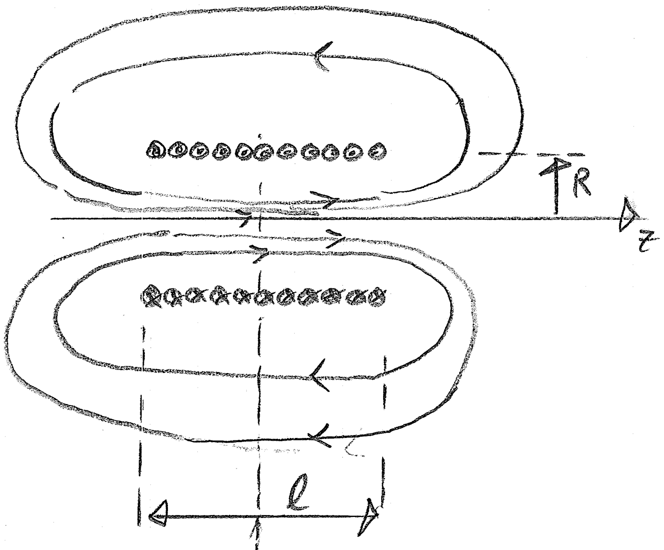
Orbit



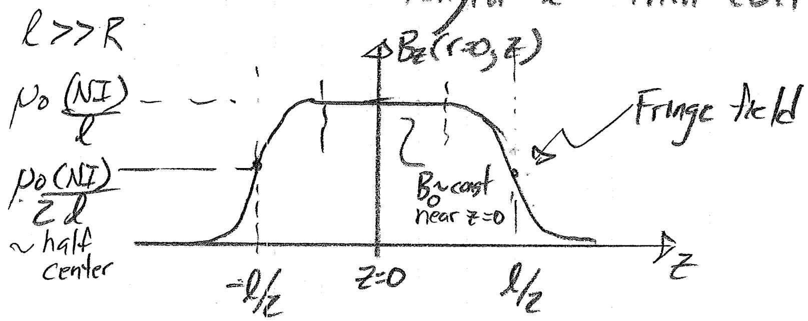
Orbit is Helix in 3D.

So how does a solenoid focus?!

Real solenoid has ends:



(NI) Amp turns in axial length ℓ thin coil



Approximate for small r: $B_z = B_0(z)$

Linear Optics Fields

$$\boxed{\begin{aligned} B_r &= -\tfrac{1}{2} B_0'(z)\, r + \cdots\cdots \\ B_z &= B_0(z) + \underbrace{\cdots\cdots}_{\text{Neglect}} \end{aligned}}$$

By Symmetry

$$\vec{B} = B_r(r,z)\,\hat{r} + B_z(r,z)\,\hat{z}$$

E&M analysis shows that

Biot-Savart: see Jackson: Classical Electromagnetic Theory

$$B_0(z) \equiv B_z(r=0,z)$$

$$= \frac{\mu_0 (NI)}{2\ell}\left[\frac{(z+\ell/2)}{\sqrt{(z+\ell/2)^2+R^2}} - \frac{(z-\ell/2)}{\sqrt{(z-\ell/2)^2+R^2}}\right]$$

straightforward to calculate: superimpose center field of current loops.

$$\nabla\cdot\vec{B} = 0 \quad\Rightarrow\quad \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0$$

$$\nabla\times\vec{B} = 0 \quad\Rightarrow\quad \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\theta} = 0$$

$$\overset{\nabla\cdot\vec{B}=0}{\Rightarrow} \quad \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + B_0'(z) = 0$$

$$\Rightarrow\quad B_r = -\frac{1}{2} B_0'(z)\, r$$

Same formulas work for iron yoke solenoid; but iron changes $B_0(z)$ function from vacuum formula.

Particles enter solenoid from outside where $\vec{B}=0$. System is rotationally symmetric so expect a conserved canonical angular momentum:

$$P_\theta = [\underbrace{\vec{x}}_{\text{coord}} \times (\underbrace{\vec{P}+q\vec{A}}_{\text{canonical momentum}})] \cdot \hat{z} = \text{const}$$

$$\vec{B} = \nabla \times \vec{A} \quad : \quad \vec{A} = A_\theta \hat{\theta} \cdot \quad B_r = -\frac{\partial A_\theta}{\partial z} = -\frac{1}{2} B_0'(z)\, r$$

$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) = B_0(z)$$

$A_\theta = \frac{1}{2} B_0(z)\, r$ works.

$$\Rightarrow \quad \boxed{P_\theta = m\gamma r v_\theta + \frac{q}{2} B_0(z)\, r^2 = \text{const}}$$

← Can verify this is true from eqns of motion

- Basically shows that beam generates angular velocity while entering solenoid due to $q v_z \hat{z} \times B_r \hat{r}$ Lorentz force.
- Can argue same result from an impulse argument with Lorentz force equation.

If beam is initially unmagnetized (born outside magnetic field), then

$$P_\theta = 0 \quad \Rightarrow \quad \boxed{v_\theta = -\frac{q B_0(z)}{2 \gamma m}\, r}$$

Magnetized

$$v_\theta = \frac{P_\theta}{m\gamma r} - \frac{q B_0(z)}{2\gamma m}\, r$$

v_θ gained depends on distance from solenoid axis ($r=0$) and $\perp$ radius of gyration will be:

PS 2/ $r_\perp = \frac{\gamma m |v_\perp|}{q B_0}$:

$$\boxed{r_\perp = \frac{\gamma m |v_\theta|}{q B_0} = \frac{r}{2}}$$

We are now in a position to derive a radial eqn of motion for the solenoid and interpret it:

$$\frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B} \qquad m\gamma\ddot{\vec{x}} = q\,\dot{\vec{x}}\times\vec{B} \qquad \vec{B} \simeq -\tfrac{1}{2}B_0' r\,\hat{r} + B_0\hat{z}$$

since $|\vec{v}| = \text{const}$

$\Rightarrow$ $\gamma = \text{const}$ in magnetic field

$$\vec{x} = r\,\hat{r} + z\,\hat{z}$$

$$\dot{\vec{x}} = \dot{r}\,\hat{r} + \underbrace{(r\dot{\theta})}_{=v_\theta}\hat{\theta} + \dot{z}\,\hat{z}$$

$$\ddot{\vec{x}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\,\hat{z}$$

$\Leftarrow$

$$\hat{r} \equiv \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} \equiv -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\Rightarrow \begin{cases} \dot{\hat{r}} = [-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}]\dot{\theta} = \dot{\theta}\,\hat{\theta} \\ \dot{\hat{\theta}} = -[\cos\theta\,\hat{x} + \sin\theta\,\hat{y}]\dot{\theta} = -\dot{\theta}\,\hat{r} \end{cases}$$

Radial Components

$$m\gamma\ddot{\vec{x}} = q\,\dot{\vec{x}}\times\vec{B} \Rightarrow m\gamma(\ddot{r} - r\dot{\theta}^2) = q\,[\dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta} + \dot{z}\hat{z}]\times[-\tfrac{1}{2}B_0' r\,\hat{r} + B_0\hat{z}]\cdot\hat{r}$$

$$= q(r\dot{\theta})B_0 \quad \text{only term.}$$

$$\Rightarrow \ddot{r} - r\dot{\theta}^2 = \frac{q}{\gamma m}(r\dot{\theta})B_0$$

But $v_\theta = -\frac{qB_0}{2\gamma m} r = r\dot{\theta}$ From $P_\theta = 0 = \text{const}$

$$\Rightarrow \ddot{r} - \left(\frac{qB_0}{2\gamma m}\right)^2 r = -2\left(\frac{qB_0}{2\gamma m}\right)^2 r$$

$$\Rightarrow \ddot{r} + \left(\frac{qB_0}{2\gamma m}\right)^2 r = 0$$

$$\Rightarrow v^2\frac{d^2}{dz^2}r + \left(\frac{qB_0}{2\gamma m}\right)^2 r = 0$$

$$\boxed{\frac{d^2}{dz^2}r + \left(\frac{B_0(z)}{2(B\rho)}\right)^2 r = 0}$$

Recast using z rather than t as the independent variable:

$$\dot{\ } \equiv \frac{dz}{dt}\frac{d}{ds} = v_z\frac{d}{dz} \simeq v\frac{d}{dz}$$

$v \simeq \text{const}$

$\ddot{r} \simeq v^2\frac{d^2}{dz^2}r$ paraxial approx.

$$(B\rho) = \frac{\gamma m v}{q} = \frac{p}{q} = \text{Rigidity}$$

$$= \frac{\text{momentum}}{\text{charge}}$$

The radial eqn of motion:

$$\frac{d^2}{dz^2} r + \left(\frac{B_0(s)}{2(B\rho)}\right)^2 r = 0$$

has the form of Hill's eqn $x_j'' + K(s)x = 0$

$$x \to r \qquad K(s) \to \left(\frac{B_0(z)}{2(B\rho)}\right)^2 \qquad s \to z$$

The same transfer matrix analysis can be applied to this radial equation

$$\begin{bmatrix} r \\ r' \end{bmatrix}_z = \bar{M}(z|z_i) \begin{bmatrix} r \\ r' \end{bmatrix}_{z_i} \underset{*}{\simeq} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r \\ r' \end{bmatrix}_{z_i}$$

Thin lens approx. for short solenoid kicking orbit.

for transport through the solenoid

$$\boxed{\frac{1}{f} = K(0)\,l = \left(\frac{B_0(0)}{2(B\rho)}\right)^2 l}$$

Thin lens focal length: valid when $f >> l$.

* Need to take some avg measure K here: guess middle value

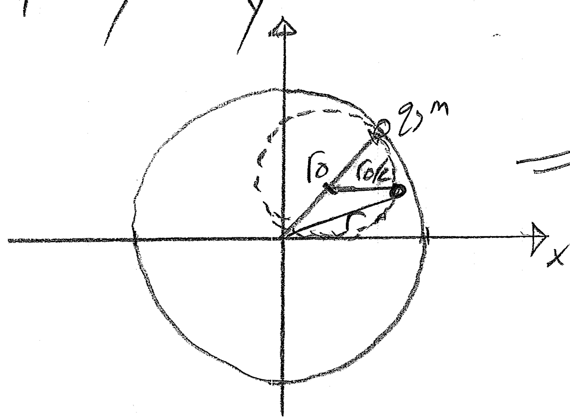
- This scaling implies that solenoid will be a <u>stronger</u> when the momentum p is small and/or when the charge $q = Qe$ is high.
  * Better for low energy beam.
- Solenoid will also focus both transverse planes simultaneously, which can reduce peak beam excursions in the machine apeture, relative to AG Quad focus
  * More compact beam envelope. .... but xy motion coupled complicating dynamics.

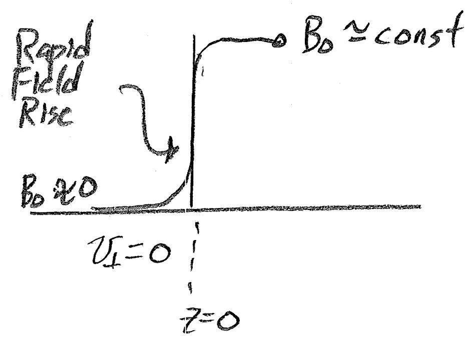
Let's further interpret the solenoid result to better understand:

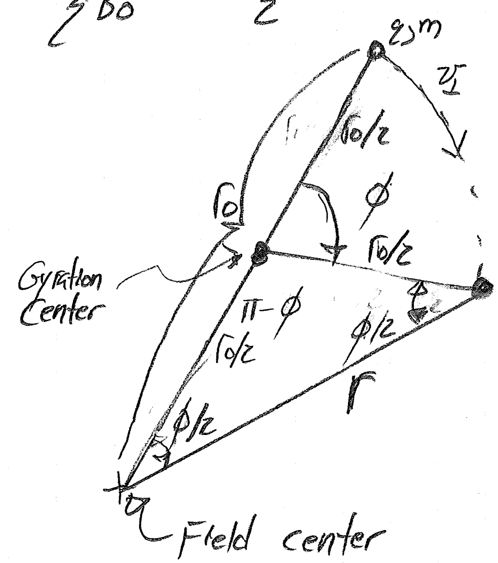
1) Particle enters at radius r_0 from field center with no initial angle.

- Will oscillate with $\omega = \frac{q B_0(z)}{2\gamma m} = \frac{\omega_c}{\gamma}$ when in central field
- Acquires angular velocity $v_\theta = -\frac{q B_0}{2\gamma m} r_0$ from impulse on fringe (P_θ conservation)
- Radius gyration will be $r_\perp = \frac{\gamma m |v_\theta|}{q B_0} = \frac{r_0}{2}$

Graphically







r_0 = initial (entry) radius

$r_0/2$ = gyration radius due $v_\perp$ generated on transit of fringe

ϕ = oscillation phase,

$= \frac{q B_0}{2\gamma m} \frac{(z-z_0)}{v}$

Law of sines:

$$\frac{r}{\sin(\pi-\phi)} = \frac{r_0/2}{\sin(\phi/2)}$$

$$r = \frac{r_0}{2}\frac{\sin(\pi-\phi)}{\sin(\phi/2)} = \frac{r_0}{2}\frac{(\overbrace{\sin\pi}^{0}\cos\phi - \overbrace{\cos\pi}^{-1}\sin\phi)}{\sin(\phi/2)} = \frac{r_0}{2}\frac{\sin\phi}{\sin(\phi/2)} = r_0\cos\left(\frac{\phi}{2}\right)$$

$$= r_0 \cos\left(\frac{q B_0}{2\gamma m}(z-z_0)\right)$$

Thus:

$$r(z) = r_0 \cos\left(\frac{q B_0 (z - z_0)}{\cancel{2\gamma} m v}\right)$$

$$\frac{d^2 r(z)}{dz^2} = -r_0 \left(-\frac{B_0}{2(B\rho)}\right)^2 \cos\left(-\frac{B_0(z-z_0)}{(B\rho)}\right) = -\left(\frac{B_0}{2(B\rho)}\right)^2 r$$

$$\boxed{\frac{d^2 r(z)}{dz^2} + \left(\frac{B_0}{2(B\rho)}\right)^2 r = 0}$$

Same result as before, but geometric argument provides interpretation to the focusing effect.

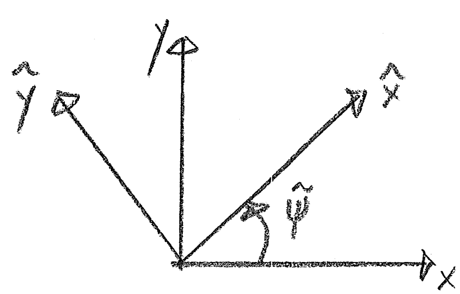
★ Fringe field going in and out of optic generate effective radial focus.

What if initial beam has asymmetries or initial canonical angular momentum? Such situations can be systematically analyzed by using a transformation to a rotating "Larmor" frame of reference. See Lund USPAS notes on Transverse Particle Dynamics Sections S2E–S2G Appendices B–D (pgs 112–132); see 03.lecture.pdf.

☆ Allows systematic analysis of much more complex problems.
☆ Or numerically advance orbits from initial conditions in solenoid field.

Summary outline: use $z \to s$

If a transformation to a rotating "Larmor" frame is applied.



$$\tilde{x} = x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$
$$\tilde{y} = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = \int_{s_i}^{s} \frac{B_0(\tilde{s})}{2(B\rho)}\, d\tilde{s} = \text{Larmor Phase}$$

Then

Cross-Coupled Solenoid equations of motion

$$B_r = -\frac{1}{2}B_0'(s)\, r$$
$$B_z = B_0(s)$$

$$x'' - \frac{B_0'(s)}{2(B\rho)}\, y - \frac{B_0(s)}{(B\rho)}\, y' = 0$$

$$y'' + \frac{B_0'(s)}{2(B\rho)}\, x + \frac{B_0(s)}{(B\rho)}\, x' = 0$$

Becomes: (Hill's equation form in rotating frame)

$$\tilde{x}'' + \kappa(s)\,\tilde{x} = 0$$
$$\tilde{y}'' + \kappa(s)\,\tilde{y} = 0$$

$$\kappa(s) = \left(\frac{B_0(s)}{2(B\rho)}\right)^2 \quad \text{Lattice Focus Function}$$

Comments

- ★ Allows analysis of arbitrary distributions of particles (no assumed symmetries outside of linear optics fields)

- ★ The formulation also works with combined axial acceleration ($\gamma\beta \neq$ const)
  - See Lund USPAS notes, Appendix A for details.
  - Important since solenoid fields often overlap acceleration gaps near sources, where solenoids are often used.

- ★ Initial conditions must be properly transformed to rotating frame to apply the formulation.

<u>$s = s_i$ Generally</u>

$$\tilde{x}(s_i) = x(s_i)$$
$$\tilde{y}(s_i) = y(s_i)$$

<u>If $B_0(s_i) = 0$</u> (outside fringe; usual case)

$$\tilde{x}'(s_i) = x'(s_i)$$
$$\tilde{y}'(s_i) = y'(s_i)$$

Caution: Angles must change if $B_0(s_i) \neq 0$.

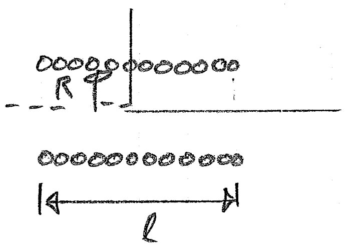
Often we want to apply formulation to a model with piecewise constant $R(s)$. For solenoids, this is conceptually more awkward since the fringe field provides the focusing. However, if reasonable equivalances are applied it is found that this procedure works surprisingly well.

Equivalance procedure: for Hard-Edge Solenoid

see Lund USPAS Appendix C

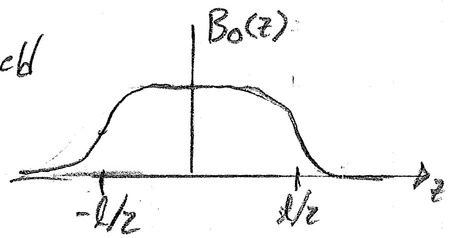
Physical

$$B_z(r=0, z) = B_0(z) = \frac{\mu_0 (NI)}{2}\left[\frac{(z+l/2)}{\sqrt{(z+l/2)^2+R^2}} - \frac{(z-l/2)}{\sqrt{(z-l/2)^2+R^2}}\right]$$

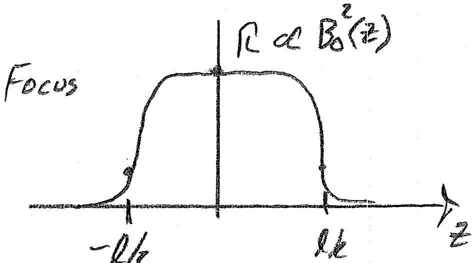


Sketch thin coil but applies to any solenoid with or without iron.

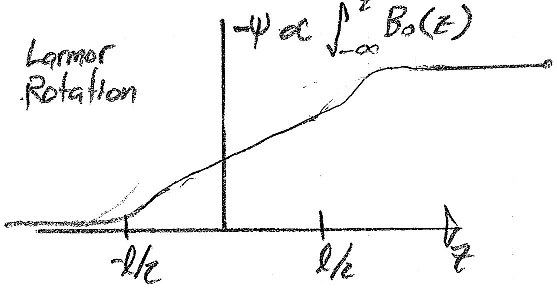
Field



Focus

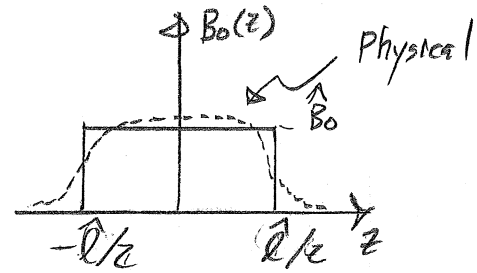


Larmor Rotation



Replace $\Longrightarrow$ with

Hard Edge Equivalent



$\hat{B}_0$ = const effective field

$\hat{l}$ = const effective length

Require equivalnce

1) Same focusing impulse $K \propto B_0^2$

$$\Rightarrow \int_{-\infty}^{\infty} B_0^2(z)\Big|_{\text{Physical}} dz = \int_{-\infty}^{\infty} B_0^2(z)\Big|_{\text{Hard Edge}} dz \equiv \hat{l}\,\hat{B}_0^2$$

2) Same Larmor rotation $\psi \propto \int_{-\infty}^{\infty} B_0(z)\,dz$

$$\Rightarrow \int_{-\infty}^{\infty} B_0(z)\Big|_{\text{Physical}} dz = \int_{-\infty}^{\infty} B_0(z)\Big|_{\text{Hard Edge}} dz = \hat{l}\,\hat{B}_0$$

1) and 2) provide two equations for $\hat{\ell}$ and $\hat{B}_0$. Solution gives:

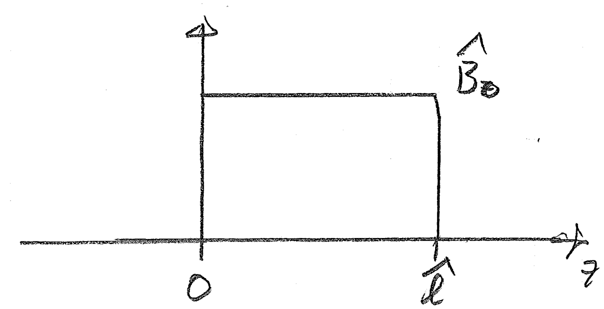
Hard Edge Equivalence

$$\boxed{\hat{\ell} = \frac{\left[\int_{-\infty}^{\infty} B_0(z)\,dz\right]^2}{\int_{-\infty}^{\infty} B_0^2(z)\,dz} \qquad \hat{B} = \frac{\int_{-\infty}^{\infty} B_0^2(z)\cdot dz}{\int_{-\infty}^{\infty} B_0(z)\,dz} \qquad B_0(z) = \text{Physical Magnet Field Function}}$$

☆ Studies find this produces surprisingly accurate results in FRIB simulations. Q. Zhao, H. He.

To solve for beam focusing properties in a hard-edge solenoid, the Larmor frame formulation can be exploited with a 4x4 transfer matrix:

$$\underbrace{\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{\ell^+}}_{\substack{\text{Lab coordinate} \\ \text{after solenoid} \\ z = \ell^+}} = \underbrace{\overline{M}(\ell^+ | 0^-)}_{\substack{\text{4x4 Matrix} \\ \text{from before} \\ \text{solenoid } (0^-) \\ \text{to end } (\ell^+)}} \cdot \underbrace{\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}_{0^-}}_{\substack{\text{lab} \\ \text{coordinate} \\ \text{before solenoid} \\ z = 0^-}}$$



Analysis shows that:

$$\bar{M}(\ell^+|0^-) = \begin{bmatrix} \cos^2\Phi & \frac{1}{2K_L}\sin(2\Phi) & \frac{1}{2}\sin\Phi & \frac{1}{K_L}\sin^2\Phi \\ -\frac{K_L}{2}\sin(2\Phi) & \cos^2\Phi & -K_L\sin^2\Phi & \frac{1}{2}\sin(2\Phi) \\ -\frac{1}{2}\sin(2\Phi) & -\frac{1}{K_L}\sin^2\Phi & \cos^2\Phi & \frac{1}{2K_L}\sin(2\Phi) \\ K_L\sin^2\Phi & -\frac{1}{2}\sin(2\Phi) & -\frac{K_L}{2}\sin(2\Phi) & \cos^2\Phi \end{bmatrix} \qquad \begin{aligned} \Phi &\equiv K_L\hat{\ell} \\ K_L &= \frac{\hat{B}_0}{2(B\rho)} \end{aligned}$$

Which is resolvable to:

$$\bar{M}(\ell^+|0^-) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K_L & 0 \\ 0 & 0 & 1 & 0 \\ K_L & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{"Thin Lens"} \\ \text{Exiting} \\ \ell^- \to \ell^+}} \cdot \underbrace{\begin{bmatrix} 1 & \frac{1}{2K_L}\sin(2\Phi) & 0 & \frac{1}{K_L}\sin^2\Phi \\ 0 & \cos(2\Phi) & 0 & \sin(2\Phi) \\ 0 & \frac{1}{K_L}\sin^2\Phi & 1 & \frac{1}{2K_L}\sin(2\Phi) \\ 1 & -\sin(2\Phi) & 0 & \cos(2\Phi) \end{bmatrix}}_{\substack{\text{Body} \\ 0^+ \to \ell^-}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & K_L & 0 \\ 0 & 0 & 1 & 0 \\ -K_L & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{"Thin Lens"} \\ \text{Entering} \\ 0^- \to 0^+}}$$

☆ Solenoid clearly has more complicated focusing than dipoles and quadrupole magnets – in spite of simple, symmetrical field structure.

☆ Topic often not covered in texts, or covered poorly. We use them a lot at MSU and FRIB. Appropriate to go into details. see 06.lecture.pdf from Transverse Particle Dynamics USPAS notes.