12. Acceleration and Normalized Emittance*

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S10: Acceleration and Normalized Emittance
S10A: Introduction

If the beam is accelerated longitudinally in a linear focusing channel, the *x*-particle equation of motion is:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = \left[-\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}\right]$$

Analogous equation holds in *y*

Neglects:
- Nonlinear applied focusing fields
- Momentum spread effects

In this class we will neglect space-charge with: $[\cdots] \simeq 0$

Comments:
- $\gamma_b,\ \beta_b$ are regarded as prescribed functions of *s* set by the acceleration schedule of the machine/lattice
- Variations in $\gamma_b,\ \beta_b$ due to acceleration must be included in and/or compensated by adjusting the strength of the optics via optical parameters contained in $\kappa_x,\ \kappa_y$ to maintain lattice quasi-periodicity
  - Example: for quadrupole focusing adjust field gradients (see: S2)

Acceleration Factor: Characteristics of Relativistic Factor

$$\gamma_b\beta_b \simeq \begin{cases}\gamma_b, & \text{Ultra Relativistic Limit}\\ \beta_b, & \text{Nonrelativistic Limit}\end{cases}$$

$$\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

Beam/Particle Kinetic Energy:

$$\mathcal{E}_b(s) = (\gamma_b - 1)mc^2 = \text{Beam Kinetic Energy}$$

- Function of s specified by Acceleration schedule for transverse dynamics
- See S11 for calculation of $\mathcal{E}_b$ and $\gamma_b\beta_b$ from longitudinal dynamics and later lectures on Longitudinal Dynamics

Approximate energy gain from average gradient:

$$\mathcal{E}_b \simeq \mathcal{E}_i + G(s - s_i)$$

$$\mathcal{E}_i = \text{const} = \text{Initial Energy}$$
$$G = \text{const} = \text{Average Gradient}$$

- Real energy gain will be rapid when going through discreet acceleration gaps

$$\mathcal{E}_b \simeq \begin{cases}\gamma_b mc^2, & \text{Ultra Relativistic Limit}, \gamma_b \gg 1\\ \frac{1}{2}m\beta_b^2c^2, & \text{Nonrelativistic Limit}, |\beta_b| \ll 1\end{cases}$$

Comments Continued:
- In typical accelerating systems, changes in $\gamma_b\beta_b$ are slow and the fractional changes in the orbit induced by acceleration are small
  - Exception near an injector since the beam is often not yet energetic
- The acceleration term:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} > 0$$

will act to **damp particle oscillations** (see following slides for motivation)

Even with acceleration, we will find that there is a Courant-Snyder invariant (normalized emittance) that is valid in an analogous context as in the case without acceleration provided phase-space coordinates are chosen to compensate for the damping of particle oscillations

Identify relativistic factor with average gradient energy gain:

Ultra Relativistic Limit: $\gamma_b \gg 1, \quad \beta_b \simeq 1$

$$\gamma_b \simeq \frac{\mathcal{E}_b}{mc^2} = \frac{\mathcal{E}_i}{mc^2} + \frac{G}{mc^2}(s - s_i)$$

$$\Longrightarrow \quad \boxed{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} \simeq \frac{1}{\frac{\mathcal{E}_i}{G} + (s - s_i)} \sim \frac{1}{s - s_i}}$$

Nonrelativistic Limit: $|\beta_b| \ll 1, \quad \gamma_b \simeq 1$

$$\beta_b \simeq \sqrt{2\frac{\mathcal{E}_b}{mc^2}} = \sqrt{2\frac{\mathcal{E}_i}{mc^2} + 2\frac{G}{mc^2}(s - s_i)}$$

$$\Longrightarrow \quad \boxed{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1/2}{\frac{\mathcal{E}_i}{G} + (s - s_i)} \sim \frac{1}{2(s - s_i)}}$$

- Expect Relativistic and Nonrelativistic motion to have similar solutions
  - Parameters for each case will be quite different

/// Aside: Acceleration and Continuous Focusing Orbits with $\kappa_x = k_{\beta 0}^2 = \text{const}$

Assume relativistic motion and negligible space-charge:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} = \frac{1}{(\frac{\mathcal{E}_i}{G} - s_i) + s} \qquad \frac{\partial \phi}{\partial x} \simeq 0$$

Then the equation of motion $x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = 0$ reduces to:

$$\boxed{x'' + \frac{1}{(\frac{\mathcal{E}_i}{G} - s_i) + s}x' + k_{\beta 0}^2 x = 0}$$

This equation is the equation of a Bessel Function of order zero:

$$\frac{d^2x}{d\xi^2} + \frac{1}{\xi}\frac{dx}{d\xi} + x = 0 \qquad \xi = k_{\beta 0}s + k_{\beta 0}\left(\frac{\mathcal{E}_i}{G} - s_i\right)$$

$$\xi' = k_{\beta 0}$$

$$x = C_1 J_0(\xi) + C_2 Y_0(\xi)$$

$$x' = -C_1 k_{\beta 0} J_1(\xi) - C_2 k_{\beta 0} Y_1(\xi)$$

$dJ_0(x)/dx = -J_1(x)$ and same for Y_0

$C_1 = \text{const} \quad C_2 = \text{const}$

$J_n =$ Order n Bessel Func (1st kind)

$Y_n =$ Order n Bessel Func (2nd kind)

Solving for the constants in terms of the particle initial conditions:

$$\begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} J_0(\xi_i) & Y_0(\xi_i) \\ -k_{\beta 0}J_1(\xi_i) & -k_{\beta 0}Y_1(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$x_i \equiv x(s = s_i) \qquad \xi_i \equiv k_{\beta 0}\frac{\mathcal{E}_i}{G} = \xi(s = s_i)$$
$$x_i' \equiv x'(s = s_i)$$

Invert matrix to solve for constants in terms of initial conditions:

$$\Longrightarrow \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -k_{\beta 0}Y_1(\xi_i) & -Y_0(\xi_i) \\ k_{\beta 0}J_1(\xi_i) & J_0(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

$$\Delta \equiv k_{\beta 0}[Y_0(\xi_i)J_1(\xi_i) - J_0(\xi_i)Y_1(\xi_i)]$$

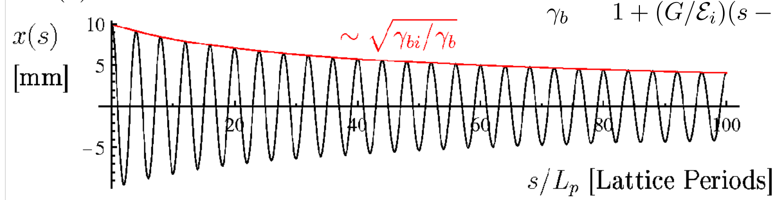
Comments:

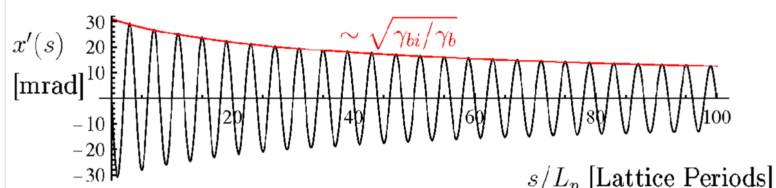
- Bessel functions behave like *damped harmonic oscillators*
  - See texts on Mathematical Physics or Applied Mathematics
- Nonrelativistic limit solution is *not* described by a Bessel Function solution
  - The coefficient in the damping term $\propto x'$ has a factor of 2 difference, preventing exact Bessel function form
  - Properties of solution will be similar though (similar special function)

Using this solution, plot the orbit for (contrived parameters for illustration only):

$$k_{\beta 0} = \frac{\sigma_0}{L_p} \qquad \sigma_0 = 90^\circ/\text{Period} \qquad \mathcal{E}_i = 1000\ \text{MeV}$$
$$L_p = 0.5\ \text{m} \qquad G = 100\ \text{MeV/m}$$
$$x(0) = 10\ \text{mm} \qquad s_i = 0$$
$$x'(0) = 0\ \text{mrad}$$

$$\frac{\gamma_{bi}}{\gamma_b} = \frac{1}{1 + (G/\mathcal{E}_i)(s - s_i)}$$





- Solution shows damping: phase volume scaling $\sim 1/(\gamma_b\beta_b) \simeq 1/\gamma_b$ ///

S10B: Transformation to Normal Form

"Guess" transformation to apply motivated by conjugate variable arguments

$$\tilde{x} \equiv \sqrt{\gamma_b \beta_b} x$$

Here we reuse tilde variables to denote a transformed quantity we choose to look like something familiar from simpler contexts

Then:

$$x = \frac{1}{\sqrt{\gamma_b \beta_b}} \tilde{x}$$

$$x' = \frac{1}{\sqrt{\gamma_b \beta_b}} \tilde{x}' - \frac{1}{2} \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^{3/2}} \tilde{x}$$

$$x'' = \frac{1}{\sqrt{\gamma_b \beta_b}} \tilde{x}'' - \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)^{3/2}} \tilde{x}' + \left[\frac{3}{4} \frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^{5/2}} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)^{3/2}} \right] \tilde{x}$$

The inverse phase-space transforms will also be useful later:

$$\tilde{x} = \sqrt{\gamma_b \beta_b} x$$

$$\tilde{x}' = \sqrt{\gamma_b \beta_b} x' + \frac{1}{2} \frac{(\gamma_b \beta_b)'}{\sqrt{(\gamma_b \beta_b)}} x$$

Applying these results, the particle *x*- equation of motion with acceleration becomes:

$$\tilde{x}'' + \left[\kappa_x + \frac{1}{4} \frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^2} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)} \right] \tilde{x} = -\frac{q}{m \gamma_b^2 \beta_b c^2} \frac{\partial \phi}{\partial \tilde{x}}$$

Note:

- Factor of $\gamma_b \beta_b$ difference from untransformed expression in the space-charge coupling coefficient

It is instructive to also transform the Possion equation associated with the space-charge term:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{\rho}{\epsilon_0}$$

Transform:

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial \tilde{x}}{\partial x} \frac{\partial}{\partial \tilde{x}} \right) \left(\frac{\partial \tilde{x}}{\partial x} \frac{\partial}{\partial \tilde{x}} \right) = \gamma_b \beta_b \frac{\partial^2}{\partial \tilde{x}^2}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial \tilde{y}}{\partial y} \frac{\partial}{\partial \tilde{y}} \right) \left(\frac{\partial \tilde{y}}{\partial y} \frac{\partial}{\partial \tilde{y}} \right) = \gamma_b \beta_b \frac{\partial^2}{\partial \tilde{y}^2}$$

Using these results, Poisson's equation becomes:

$$\left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2} \right) \phi = -\frac{\rho}{\gamma_b \beta_b \epsilon_0}$$

Or defining a transformed potential $\tilde{\phi}$

$$\tilde{\phi} = \gamma_b \beta_b \phi$$

$$\left(\frac{\partial^2}{\partial \tilde{x}^2} + \frac{\partial^2}{\partial \tilde{y}^2} \right) \tilde{\phi} = -\frac{\rho}{\epsilon_0}$$

Applying these results, the *x*-equation of motion with acceleration becomes:

$$\tilde{x}'' + \left[\kappa_x + \frac{1}{4} \frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^2} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)} \right] \tilde{x} = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \tilde{\phi}}{\partial \tilde{x}}$$

- Usual form of the space-charge coefficient with $\gamma_b^3 \beta_b^2$ rather than $\gamma_b^2 \beta_b$ is restored when expressed in terms of the transformed potential $\tilde{\phi}$

An additional step can be taken to further stress the correspondence between the transformed system with acceleration and the untransformed system in the absence of acceleration.

Denote an effective focusing strength:

$$\tilde{\kappa}_x \equiv \kappa_x + \frac{1}{4} \frac{(\gamma_b \beta_b)'^2}{(\gamma_b \beta_b)^2} - \frac{1}{2} \frac{(\gamma_b \beta_b)''}{(\gamma_b \beta_b)}$$

$\tilde{\kappa}_x$ incorporates acceleration terms beyond γ_b, β_b factors already included in the definition of κ_x (see: S2):

$$\kappa_x = \begin{cases} \frac{qG}{m \gamma_b \beta_b^2 c^2}, & G = -\partial E_x^a / \partial x = \partial E_y^a / \partial y = \text{Electric Quad. Grad.} \\ \frac{qG}{m \gamma_b \beta_b c}, & G = \partial B_x^a / \partial y = \partial B_y^a / \partial x = \text{Magnetic Quad. Grad.} \\ \frac{q B_{z0}}{4 m \gamma_b^2 \beta_b^2 c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

The transformed equation of motion with acceleration then becomes:

$$\tilde{x}'' + \tilde{\kappa}_x \tilde{x} = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \tilde{\phi}}{\partial \tilde{x}}$$

The transformed equation with acceleration has the same form as the equation in the absence of acceleration. If space-charge is negligible ($\partial\phi/\partial\mathbf{x}_\perp \simeq 0$) we have:

Accelerating System $\Longrightarrow$ Non-Accelerating System

$$\tilde{x}'' + \tilde{\kappa}_x \tilde{x} = 0 \quad\Longrightarrow\quad x'' + \kappa_x x = 0$$

Therefore, *all previous analysis* on phase-amplitude methods and Courant-Snyder invariants associated with Hill's equation in *x-x*' phase-space can be immediately applied to $\tilde{x} - \tilde{x}'$ phase-space for an accelerating beam

$$\left(\frac{\tilde{x}}{\tilde{w}_x}\right)^2 + (\tilde{w}_x\tilde{x}' - \tilde{w}'_x\tilde{x})^2 = \tilde{\epsilon} = \text{const}$$

$$\tilde{w}''_x + \tilde{\kappa}_x\tilde{w}_x - \frac{1}{\tilde{w}_x^3} = 0$$

$$\tilde{w}_x(s + L_p) = \tilde{w}_x(s)$$

$$\pi\tilde{\epsilon} = \text{Area traced by orbit} = \text{const in } \tilde{x}\text{-}\tilde{x}' \text{ phase-space}$$

- Focusing field strengths need to be adjusted to maintain periodicity of $\tilde{\kappa}_x$ in the presence of acceleration
  - Not possible to do exactly, but can be approximate for weak acceleration

S10C: Phase Space Relation Between Transformed and UnTransformed Systems

It is instructive to relate the transformed phase-space area in tilde variables to the usual *x-x*' phase area:

$$d\tilde{x} \otimes d\tilde{x}' = |J| dx \otimes dx'$$

where J is the Jacobian:

$$J \equiv \det \begin{bmatrix} \frac{\partial\tilde{x}}{\partial x}, & \frac{\partial\tilde{x}}{\partial x'} \\ \frac{\partial\tilde{x}'}{\partial x} & \frac{\partial\tilde{x}'}{\partial x'} \end{bmatrix}$$

$$= \det \begin{bmatrix} \sqrt{\gamma_b\beta_b} & 0 \\ \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b}} & \sqrt{\gamma_b\beta_b} \end{bmatrix} = \gamma_b\beta_b$$

Inverse transforms derived in S10B:

$$\tilde{x} = \sqrt{\gamma_b\beta_b}x$$

$$\tilde{x}' = \sqrt{\gamma_b\beta_b}x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b)}}x$$

Thus:

$$d\tilde{x} \otimes d\tilde{x}' = \gamma_b\beta_b \; dx \otimes dx'$$

Based on this area transform, if we define the (instantaneous) phase space area of the orbit trance in *x-x*' to be $\pi\epsilon_x$ "regular emittance", then this emittance is related to the "normalized emittance" $\tilde{\epsilon}_x$ in $\tilde{x} - \tilde{x}'$ phase-space by:

$$\tilde{\epsilon}_x = \gamma_b\beta_b\epsilon_x \equiv \text{Normalized Emittance} \equiv \epsilon_{nx}$$

- Factor $\gamma_b\beta_b$ compensates for acceleration induced damping in particle orbits
- Normalized emittance is very important in design of lattices to transport accelerating beams
  - Typically applied to measure beam quality when accelerating
  - Designs usually made assuming conservation of normalized emittance
- ϵ emittance measured in *x-x*' phase-space is often called "geometric emittance" to help distinguish from normalized emittance ϵ_x measured in $\tilde{x}$-$\tilde{x}'$
- The "geometric emittance" determines the beam extent and focusability with the betatron function $(x_{\max} = \sqrt{\epsilon_x\beta_x})$ so damping of ϵ_x with acceleration with conserved norm emittance $(\tilde{\epsilon}_x = \text{const})$ improves focusability on target
  - To extent $\tilde{\epsilon}_x$ grows, the improvement is degraded

S11: Calculation of Acceleration Induced Changes in gamma and beta

S11A: Introduction

The transverse particle equation of motion with acceleration was derived in a Cartesian system by approximating (see: S1):

$$\frac{d}{dt}\left(m\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq q\mathbf{E}^a_\perp + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_\perp + qB^a_z\mathbf{v}_\perp \times \hat{\mathbf{z}} - q\frac{1}{\gamma_b^2}\frac{\partial\phi}{\mathbf{x}_\perp}$$

using

$$m\frac{d}{dt}\left(\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq m\gamma_b\beta_b^2c^2\left[\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp\right]$$

to obtain:

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp \times \hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

To integrate this equation, we need the variation of β_b and $\gamma_b = 1/\sqrt{1-\beta_b^2}$ as a function of s. For completeness here, we briefly outline how this can be done by analyzing longitudinal equations of motion. More details can be found in lectures to follow on Longitudinal Dynamics.

S11B: Solution of Longitudinal Equation of Motion

Changes in $\gamma_b\beta_b$ are calculated from the longitudinal particle equation of motion:

- See equation at end of S1D

$$\frac{d}{dt}\left(m\gamma\frac{dz}{dt}\right) \simeq qE_z^a - q(v_x B_y^a - v_y B_x^a) - q\frac{\partial\phi}{\partial z}$$

Term 1 Term 2 Term 3 Neglect Rel to Term 2

Using steps similar to those in S1, we approximate terms:

Term 1: $\frac{d}{dt}\left(\gamma\frac{dz}{dt}\right) \simeq c^2\beta_b(\gamma_b\beta_b)'$ $\qquad \frac{dz}{dt} = v_z \simeq \beta_b c \qquad \gamma \simeq \gamma_b$

$\qquad \frac{d}{dt} \simeq \beta_b c\frac{d}{ds}$

Term 2: $\frac{q}{m}E_z^a \simeq -\frac{q}{m}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}$

ϕ^a is a quasi-static approximation accelerating potential (see next pages)

Term 3: $-q(v_x B_y^a - v_y B_x^a) = -q\left(\frac{dx}{dt}B_y^a - \frac{dy}{dt}B_x^a\right) \simeq 0$

- Transverse magnetic fields typically only weakly change particle energy and terms can typically be neglected relative to others

The longitudinal particle equation of motion for $\gamma_b,\ \beta_b$ then reduces to:

$$\beta_b(\gamma_b\beta_b)' \simeq -\frac{q}{mc^2}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}$$

Calculate:

$$\gamma_b' = \left(\frac{1}{\sqrt{1-\beta_b^2}}\right)' = \frac{\beta_b\beta_b'}{(1-\beta_b^2)^{3/2}} = \gamma_b^3\beta_b\beta_b'$$

First apply chain rule, then use the result above twice to simplify results:

$$\Longrightarrow \quad \beta_b(\gamma_b\beta_b)' = \beta_b^2\gamma_b' + \gamma_b\beta_b\beta_b'$$
$$= \beta_b^3\gamma_b^3\beta_b' + \gamma_b\beta_b\beta_b' = (1+\gamma_b^2\beta_b^2)\gamma_b\beta_b\beta_b' = \gamma_b^3\beta_b\beta_b'$$
$$= \gamma_b'$$

Giving:

$$\gamma_b' = -\frac{q}{mc^2}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}$$

Which can then be integrated to obtain:

$$\gamma_b = -\frac{q}{mc^2}\phi^a(r=0, z=s) + \text{const}$$

We denote the on-axis accelerating potential as:

$$V(s) \equiv \phi^a(x=y=0, z=s)$$

- Can represent RF or induction accelerating gap fields
  See: Longitudinal Dynamics lectures for more details

Using this and setting $\gamma_b(s=s_i) = \gamma_{bi}$ gives for the gain in axial kinetic energy $\mathcal{E}_b$ and corresponding changes in $\gamma_b,\ \beta_b$ factors:

$$\mathcal{E}_b = (\gamma_b - 1)mc^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$
$$\gamma_b = 1 + \mathcal{E}_{bi}/(mc^2) \qquad \mathcal{E}_{bi} = (\gamma_{bi}-1)mc^2$$
$$\beta_b = \sqrt{1-1/\gamma_b^2}$$

These equations can be solved for the consistent variation of $\gamma_b(s),\ \beta_b(s)$ to integrate the transverse equations of motion:

$$\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}} - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

Nonrelativistic limit results

In the nonrelativistic limit:

$$\gamma_b \simeq 1 + \frac{1}{2}\beta_b^2 \qquad \beta_b^2 \ll 1 \qquad \mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2 c^2$$

and the previous (relativistic valid) energy gain formulas reduce to:

$$\mathcal{E}_b \simeq \frac{1}{2}m\beta_b^2 c^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$

$$\gamma_b \simeq 1 \qquad \mathcal{E}_{bi} = \frac{1}{2}m\beta_{bi}^2 c^2$$

$$\beta_b = \sqrt{\frac{2\mathcal{E}_b}{mc^2}}$$

Using this result, in the nonrelativistic limit we can take in the transverse particle equation of motion:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1}{2}\frac{\mathcal{E}_b'}{\mathcal{E}_b} = -\frac{1}{2}\frac{qV'(s)}{q[V(s_i) - V(s)] + \mathcal{E}_{bi}}$$

Ultra-relativistic limit results

In the ultra-relativistic limit:

$$\gamma_b \gg 1 \qquad \beta_b \simeq 1 \qquad \mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \gamma_b mc^2$$

and the previous (relativistic valid) energy gain formulas reduce to:

$$\mathcal{E}_b \simeq \gamma_b mc^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$

$$\beta_b \simeq 1$$

Using this result, in the ultra-relativistic limit we can take in the transverse particle equation of motion:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} = \frac{\mathcal{E}_b'}{\mathcal{E}_b} = -\frac{qV'(s)}{q[V(s_i) - V(s)] + \mathcal{E}_{bi}}$$

- Same form as NR limit expression with only a factor of ½ difference; see also S10A

S11C: Longitudinal Solution via Energy Gain

An alternative analysis of the particle energy gain carried out in S11B can be illuminating. In this case we start from the exact Lorentz force equation with time as the independent variable for a particle moving in the full electromagnetic field:

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\vec{\beta}c \times \mathbf{B}$$

$$\mathbf{p} \equiv \gamma m\vec{\beta}c \qquad \gamma \equiv 1/\sqrt{1 - \vec{\beta}\cdot\vec{\beta}}$$

Comments:
- Formulation exact in context of classical electrodynamics
- γ, $\vec{\beta}$ not expanded
- $\mathbf{E}$, $\mathbf{B}$ electromagnetic

Dotting $mc\vec{\beta}$ into this equation:

$$mc\vec{\beta}\cdot\frac{d}{dt}(c\gamma\vec{\beta}) = qc\vec{\beta}\cdot\mathbf{E} + qc\vec{\beta}\cdot\underbrace{[c\vec{\beta}\times\mathbf{B}]}_{\to 0}$$

[1] [2]

$$\left[\vec{\beta}\cdot\vec{\beta}\dot{\gamma}\right] + \left[\gamma\vec{\beta}\cdot\dot{\vec{\beta}}\right] = \frac{q}{mc}\vec{\beta}\cdot\mathbf{E}$$

Then

$$\gamma \equiv (1 - \vec{\beta}\cdot\vec{\beta})^{-1/2}$$

Gives:

[1]: $\left[\vec{\beta}\cdot\vec{\beta}\right] = 1 - 1/\gamma^2$ [2]: $\left[\vec{\beta}\cdot\dot{\vec{\beta}}\right] = \dot{\gamma}/\gamma^3$

Inserting these factors:

$$(1 - 1/\gamma^2)\dot{\gamma} + \dot{\gamma}/\gamma^2 = \frac{q}{mc^2}\vec{\beta}\cdot\mathbf{E}$$

or:

$$\dot{\gamma} = \frac{q}{mc}\vec{\beta}\cdot\mathbf{E}$$

Equivalently: $\mathcal{E} = (\gamma - 1)mc^2$

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\left[(\gamma - 1)mc^2\right] = qc\vec{\beta}\cdot\mathbf{E}$$

- Only the electric field changes the kinetic energy of a particle
- No approximations made to this point within the context of classical electrodynamics: valid for evolving $\mathbf{E}$, $\mathbf{B}$ consistent with the Maxwell equations.

Now approximating to our slowly varying and paraxial formulation:

$$\frac{d}{dt} = c\beta_z\frac{d}{ds} \qquad \beta_z \simeq \beta \simeq \beta_b \qquad \gamma \simeq \gamma_b \qquad \mathcal{E} \simeq \mathcal{E}_b = (\gamma_b - 1)mc^2$$

and approximating the axial electric field by the applied component then obtains

$$\frac{d}{ds}\mathcal{E}_b \simeq \frac{dt}{ds}\frac{d}{dt}\left[(\gamma - 1)mc^2\right] \simeq qE_z^a$$

which is the longitudinal equation of motion analyzed in S11B.

S11D: Quasistatic Potential Expansion

In the quasistatic approximation, the accelerating potential can be expanded in the axisymmetric limit as:

- See: USPAS, *Beam Physics with Intense Space-Charge*; and Reiser, *Theory and Design of Charged Particle Beams*, (1994, 2008) Sec. 3.3.
- See also: S2, Appendix D

We take:

$$\mathbf{E}^a = -\frac{\partial \phi^a}{\partial \mathbf{x}} \qquad \longleftarrow \qquad \text{from } \nabla \times \mathbf{E}^a = 0$$

Allows us to use quasistatic electrostatic approx

and apply the results of S2, Appendix D to expand ϕ^a in terms of the on-axis potential in an axisymmetric (acceleration gap) system:

$$\phi^a(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu}\phi^a(r=0,z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu} \qquad \longleftarrow \qquad \text{from } \nabla \cdot \mathbf{E}^a = 0$$

Allows us to expand field in terms of derivative of the on-axis potential

Denote for the on-axis potential

$$\phi^a(r=0,z) \equiv V(z)$$

$$\Longrightarrow \phi^a = V(z) - \frac{1}{4}\frac{\partial^2}{\partial z^2}V(z)(x^2+y^2) + \frac{1}{64}\frac{\partial^4}{\partial z^4}V(z)(x^2+y^2)^2 + \cdots$$

The longitudinal acceleration also result in a transverse focusing field

$$\mathbf{E}^a_\perp = \mathbf{E}^a_\perp|_{\text{foc}} - \frac{\partial \phi^a}{\partial \mathbf{x}_\perp}$$

$$\mathbf{E}^a_\perp|_{\text{foc}} = \text{Fields from Any Applied Focusing Optics}$$

$$-\frac{\partial \phi^a}{\partial \mathbf{x}_\perp} \simeq \frac{1}{2}\frac{\partial^2}{\partial z^2}V(z)\mathbf{x}_\perp = \text{Focusing Field from Acceleration}$$

- Results can be used to cast acceleration terms in more convenient forms. See USPAS, Beam Physics with Intense Space-Charge for more details
- RF defocusing in the quasistatic approximation can be analyzed using this formulation: we will see this in analysis that follows
- Einzel lens focusing exploits accel/de-acel cycle to make AG focusing

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/ap_2018/