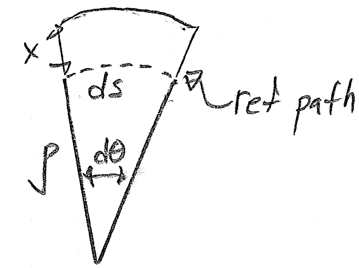
11. Momentum Compaction

The path length that a particle travels in a lattice between reference orbit coordinates s_1 and s_2 will vary with particle momentum p.

$$L = \int_{s_1}^{s_2} \sqrt{\underbrace{\left(1 + x/\rho\right)^2}_{\text{due to bend}} + \underbrace{x'^2 + y'^2}_{\text{from particle offset}}}\, ds$$



Expand for small orbit angles (paraxial approximation)

$$\Rightarrow \boxed{L \simeq \int_{s_1}^{s_2} \left(1 + \frac{x}{\rho}\right) ds} \qquad x'^2, y'^2 \ll 1$$

Neglect the relatively small contribution to x from betatron motion, but account for the larger contribution due to dispersive effects from off momentum then:

Essentially measures shift in x of orbit center with δ.

$$\boxed{\begin{aligned} x &\simeq D(s)\frac{\delta p}{p_0} = D(s)\,\delta \\ L &\simeq \int_{s_1}^{s_2} ds + \delta \int_{s_1}^{s_2} \frac{D(s)}{\rho(s)}\, ds \end{aligned}}$$

$$\boxed{\begin{aligned} L_0 &\equiv \int_{s_1}^{s_2} ds = \text{Reference orbit path length} \\ &= s_2 - s_1 \end{aligned}}$$

$$L \simeq L_0 + \delta \int_{s_1}^{s_2} \frac{D(s)}{\rho(s)}\, ds$$

Denote $\Delta L = L - L_0$ = change in path length due to off-momentum δ.

$$\Rightarrow \boxed{\Delta L = \delta \int_{s_1}^{s_2} \frac{D(s)}{\rho(s)} ds}$$

Define the <u>momentum compaction factor α_c</u> by

$$\boxed{\frac{\Delta L}{L_0} \equiv \alpha_c \frac{\delta p}{p_0} \qquad \alpha_c \equiv \frac{\int_{s_1}^{s_2} \frac{D(s)}{\rho(s)} ds}{\int_{s_1}^{s_2} ds} = \text{Momentum "Compaction" factor}}$$

☆ α_c is average of D/ρ over ideal (reference) path.

☆ α_c is a property of the lattice.

☆ For a ring, typical to take path over full ring lap reference path.

$$\boxed{\alpha_c = \frac{\oint_{ring} \frac{D(s)}{\rho(s)} ds}{\oint_{ring} ds}}$$

In this context, we <u>define</u> (will become clear soon why we do this)

$$\boxed{\alpha_c \equiv \frac{1}{\gamma_t^2} \qquad \gamma_t \equiv \text{Transition gamma.}}$$

For simple lattices: $\gamma_t \simeq$ number betatron oscillations in ring. See SY Lee pg 121

To prepare for needs to analyze longitudinal phase-focusing in RF cavities in rings (we will cover later), need to also analyze the time it takes for a particle to travel along the path of length L.

$$\boxed{\tau = \frac{L}{c\beta} \quad = \quad \text{Transit time. (assume no accel).}}$$

Here, $\beta = \frac{v}{c}$ not to be confused with C.S. betatron function.

$$\Rightarrow \ln \tau = \ln\left(\frac{L}{c\beta}\right) \xrightarrow[\tau, L, \beta \text{ vary}]{\text{Take differential}} \frac{d\tau}{\tau} = \frac{c\beta}{L}\left(\frac{dL}{c\beta} - \frac{L\,d\beta}{c\beta^2}\right)$$

Express Differential as

$$\boxed{\begin{aligned}\frac{\Delta\tau}{\tau_0} &= \frac{\Delta L}{L_0} - \frac{\Delta\beta}{\beta_0} \\ &= \alpha_c \frac{\delta p}{p_0} - \frac{\Delta\beta}{\beta_0}\end{aligned}}$$

$$\frac{\Delta L}{L_0} \equiv \alpha_c \frac{\delta p}{p_0}$$

subscript "0" denotes reference

need to calculate $\frac{\Delta\beta}{\beta_0}$ in terms of $\frac{\delta p}{p}$ to calculate $\frac{\Delta\tau}{\tau_0}$ in terms of $\frac{\delta p}{p_0}$.

Examine:

Expressions also true for reference particle (subscript 0)

$$E \equiv \gamma m c^2 \quad \ldots \quad \text{Total Energy}$$
$$p = m\gamma\beta c \quad \ldots \quad \text{Momentum}$$

$$\Rightarrow cp = \beta(\gamma m c^2) = \beta E \quad (1)$$

$$c\Delta p = \Delta\beta E + \beta\Delta E$$

$$\Rightarrow \frac{\delta p}{p_0} = \frac{\Delta\beta}{\beta_0} + \frac{\Delta E}{E_0} \quad (2)$$

Here we use Δ for differentials to avoid confusion with $\delta \equiv \frac{\delta p}{p_0}$

$$E^2 = c^2p^2 + m^2c^4$$
$$\Delta E E = c^2 p \delta p$$

$$\rightarrow \frac{\Delta E}{E_0} = \frac{c^2 p_0 \delta p}{E_0^2} = \frac{c(\underbrace{cp_0}_{=\beta_0 E_0 \ (1)})\delta p}{E_0^2} = \frac{c(\beta_0 E_0)\delta p}{E_0^2} = \frac{c\beta_0 \delta p}{(E_0)} = \frac{c\beta_0}{(\underbrace{cp_0/\beta_0}_{=c p_0/\beta_0 \ (1)})}\delta p$$
$$= \beta_0^2 \frac{\delta p}{p_0}$$

Thus from (2)

$$\frac{\Delta\beta}{\beta_0} = \frac{\delta p}{p_0} - \frac{\Delta E}{E_0} = (1 - \beta_0^2)\frac{\delta p}{p_0} = \frac{1}{\gamma_0^2}\frac{\delta p}{p_0}$$

$$\boxed{\frac{\Delta\beta}{\beta_0} = \frac{1}{\gamma_0^2}\frac{\delta p}{p_0}}$$

Using these results,

$$\frac{\Delta \tau}{\tau_0} = \frac{\Delta L}{L_0} - \frac{\Delta \beta}{\beta_0}$$

$$= \alpha_c \frac{\delta p}{p_0} - \frac{1}{\gamma_0^2}\frac{\delta p}{p_0} = \left(\alpha_c - \frac{1}{\gamma_0^2}\right)\frac{\delta p}{p_0}$$

Define:

$$\boxed{\eta_s \equiv \frac{1}{\gamma_0^2} - \alpha_c = \frac{1}{\gamma_0^2} - \frac{1}{\gamma_t^2} = \text{"Slip Factor"}}$$

Then,

$$\boxed{\frac{\Delta \tau}{\tau} = -\eta_s \cdot \frac{\delta p}{p_0}}$$

Notice that at $\gamma_0 = \gamma_t = \frac{1}{\sqrt{\alpha_c}}$ $\eta_s = 0$ corresponding to the "transition energy" of a ring.

- At this point, $\Delta\tau/\tau$ will really not be zero, but will depend on higher powers of $\delta p/p_0$.

The formula for $\Delta\tilde{T}/\tilde{T}$ is important when examining conditions to maintain synchronism with RF cavities in a ring.

Denote:

ω_r = Angular frequency to complete revolution in ring $= \dfrac{2\pi}{\tilde{T}_r}$

$\tilde{T}_r$ = revolution period in ring

$\tilde{T}_{ro}$ = design revolution period in ring

Then

$$\Delta\omega_r = \frac{-2\pi\,\Delta\tilde{T}_r}{\tilde{T}_{ro}^2} \Rightarrow \frac{\Delta\omega_r}{\omega_{ro}} = \frac{-\Delta\tilde{T}_r}{\tilde{T}_{ro}} = \eta_s \frac{\delta p}{p_0}$$

$$\boxed{\frac{\Delta\tilde{T}_r}{\tilde{T}_{ro}} = -\frac{\Delta\omega_r}{\omega_{ro}} = -\eta_s \frac{\delta p}{p_0}}$$

Case		Particle Energy		Slip Factor	Consequence
$\gamma > \gamma_t$	$\Leftrightarrow$	"Above Transition"	$\Leftrightarrow$	$\eta_s < 0$	high momentum particle takes longer time to make lap.
$\gamma < \gamma_t$	$\Leftrightarrow$	"Below Transition"	$\Leftrightarrow$	$\eta_s > 0$	high momentum particle takes shorter time to make lap.

$\therefore$ RF cavity phase control in a ring is different above and below transition energy. Synchronism changes.

★ Not surprisingly, Going through transition in an acceleration cycle can be a problem!

- When $\gamma = \gamma_t$, $\eta_s = 0$ does not really imply $\dfrac{\Delta\tilde{T}}{\Delta\tilde{T}_{ro}} = 0$. Higher order terms will matter in this case and analysis will be more difficult. (nonlinear).

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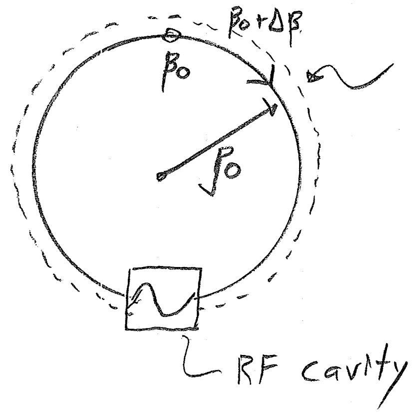
For a ring, conceptually:



Momentum deviation makes path length deviation.

We will take this into account using

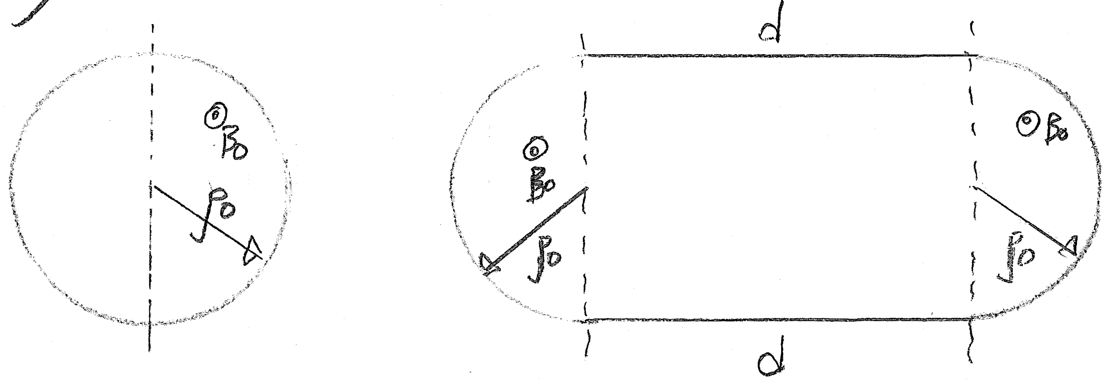
$$\frac{\Delta T_r}{T_{r0}} = -\eta_s \frac{\delta p}{p_0}$$

when studying RF acceleration in a ring.

Note $\eta_s = \frac{1}{\gamma_0^2} - \alpha_c$

with α_c a property of the lattice.

In the problem sets, you will calculate the slip factor for a continuous ring and a racetrack with continuous bends seperated by drifts of length d. The transition γ, γ_t, can be varied by altering the straight section length d.



Further examples will be explored in the simulation labs.