# 02. Multipole Fields*

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# Particle Equations of Motion

## Introduction: The Lorentz Force Equation

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\boxed{\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i,t)\right]}$$

$m_i, \quad q_i$ .... particle mass, charge

$i =$ particle index

$t =$ time

$\mathbf{x}_i(t)$ .... particle coordinate

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

| | Total | | Applied | | Self |
|---|---|---|---|---|---|
| Electric Field: | $\mathbf{E}(\mathbf{x},t)$ | $=$ | $\mathbf{E}^a(\mathbf{x},t)$ | $+$ | $\mathbf{E}^s(\mathbf{x},t)$ |
| Magnetic Field: | $\mathbf{B}(\mathbf{x},t)$ | $=$ | $\mathbf{B}^a(\mathbf{x},t)$ | $+$ | $\mathbf{B}^s(\mathbf{x},t)$ |

The electric ($\mathbf{E}^a$) and magnetic ( $\mathbf{B}^a$ ) fields satisfy the Maxwell Equations. The linear structure of the Maxwell equations can be exploited to resolve the field into Applied and Self-Field components:

$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$

$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied Fields (often quasi-static $\partial/\partial t \simeq 0$ ) $\mathbf{E}^a, \ \mathbf{B}^a$

Generated by elements in lattice

$$\nabla \cdot \mathbf{E}^a = \frac{\rho^a}{\epsilon_0} \qquad \nabla \times \mathbf{B}^a = \mu_0 \mathbf{J}^a + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^a$$

$$\nabla \times \mathbf{E}^a = -\frac{\partial}{\partial t}\mathbf{B}^a \qquad \nabla \cdot \mathbf{B}^a = 0$$

$$\begin{aligned} \rho^a &= \text{applied charge density} \\ \mathbf{J}^a &= \text{applied current density} \end{aligned} \qquad \frac{1}{\mu_0 \epsilon_0} = c^2$$

$$+ \text{ Boundary Conditions on } \mathbf{E}^a \text{ and } \mathbf{B}^a$$

- Boundary conditions depend on the total fields $\mathbf{E}, \ \mathbf{B}$ and if separated into Applied and Self-Field components, care can be required
- System often solved as static boundary value problem and source free in the vacuum transport region of the beam

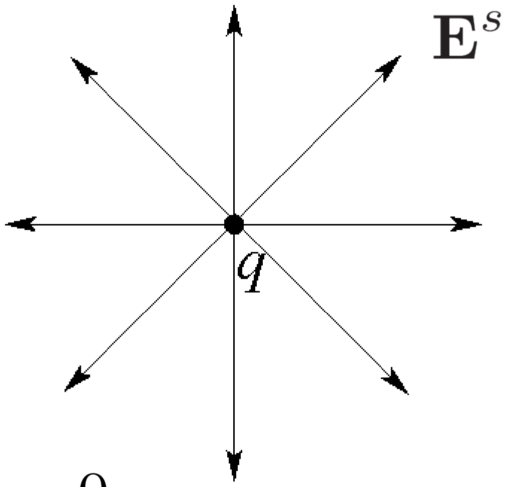
# Self fields

Self-fields are generated by the distribution of beam particles:

- Charges
- Currents

Particle at Rest

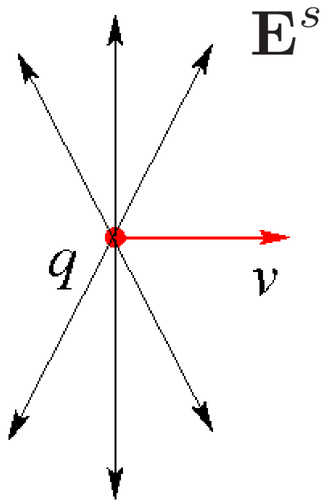
(pure electrostatic)

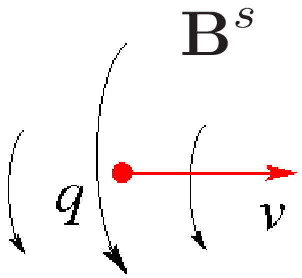


$$\mathbf{B}^s = 0$$

Particle in Motion

Obtain from
Lorentz boost
of rest-frame field:
see Jackson,
*Classical*
*Electrodynamics*





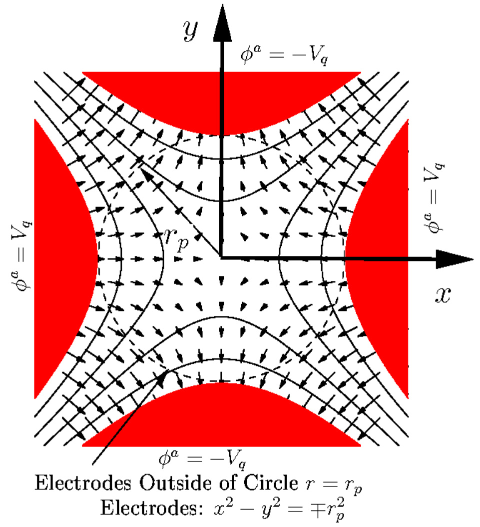
- Superimpose for all particles in the beam distribution
- Accelerating particles also radiate

We will neglect all self-field in this section: assume low intensity

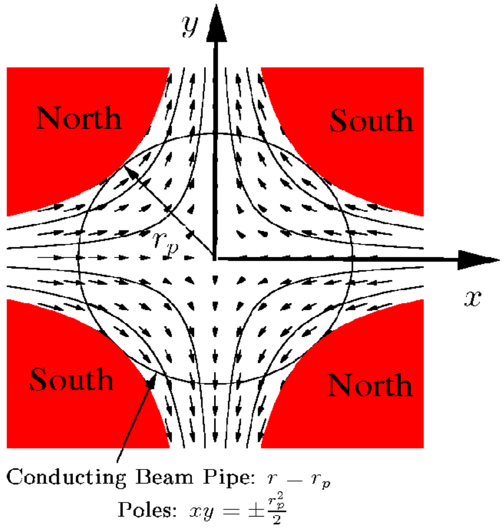
# Applied Fields used to Focus, Bend, and Accelerate Beam

## Transverse optics for focusing:

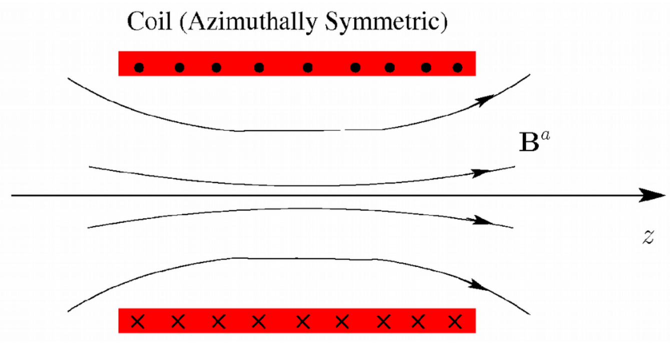
**Electric Quadrupole**



**Magnetic Quadrupole**

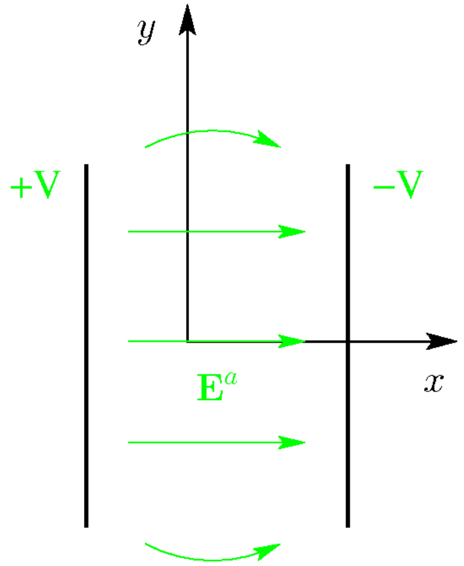


**Solenoid**

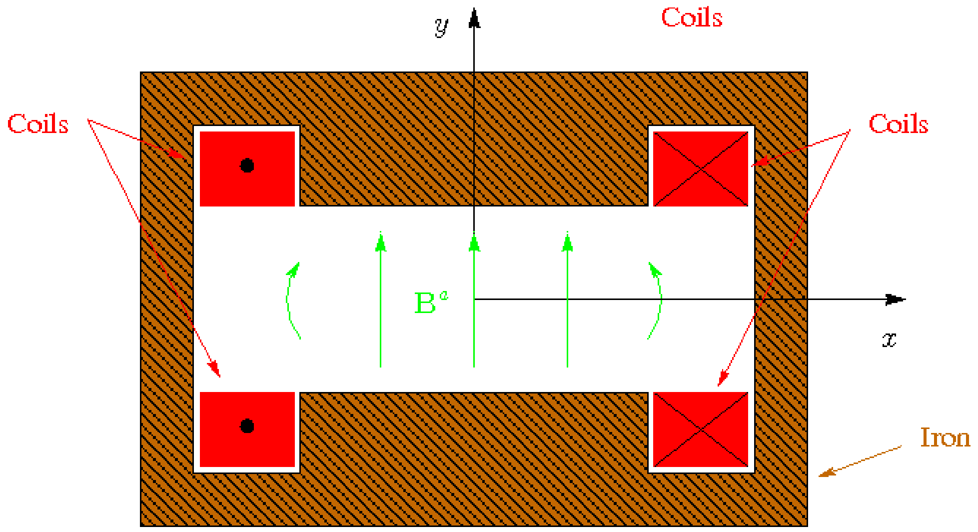


## Dipole Bends:

**Electric** $x$-direction bend

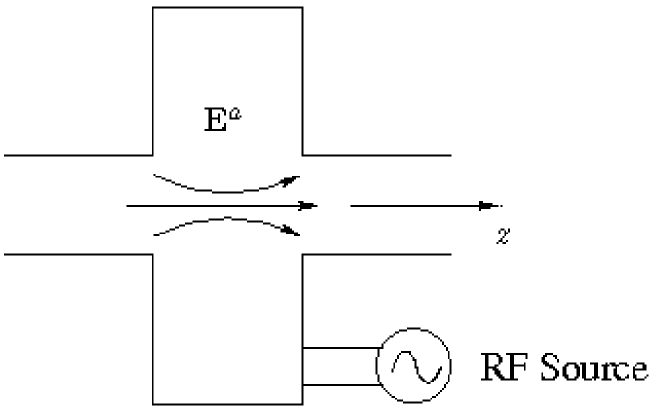


**Magnetic** $x$-direction bend

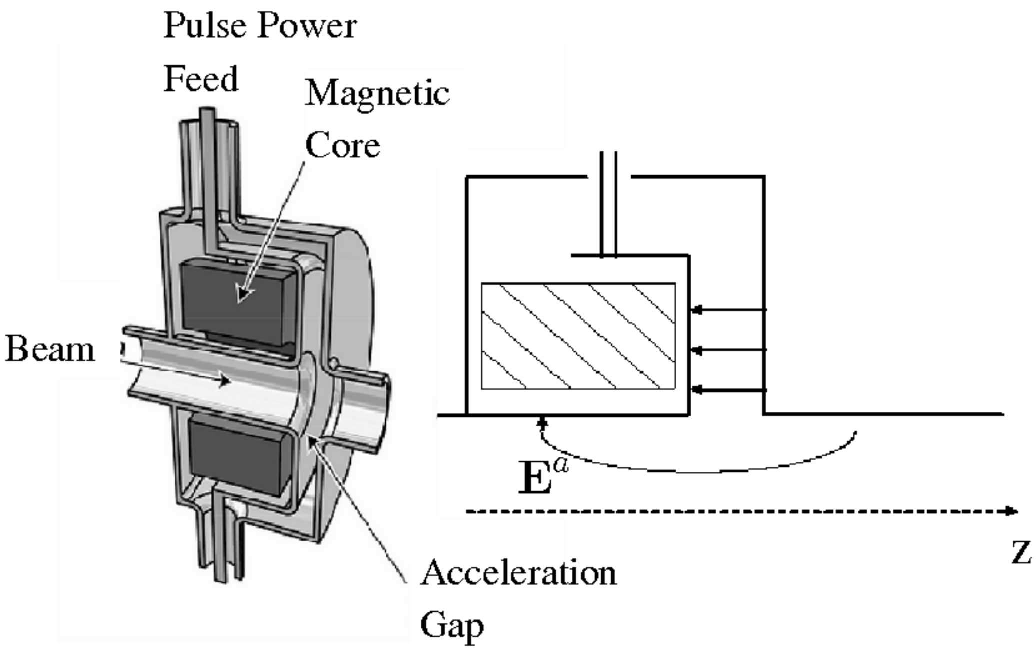


## Longitudinal Acceleration:

### RF Cavity

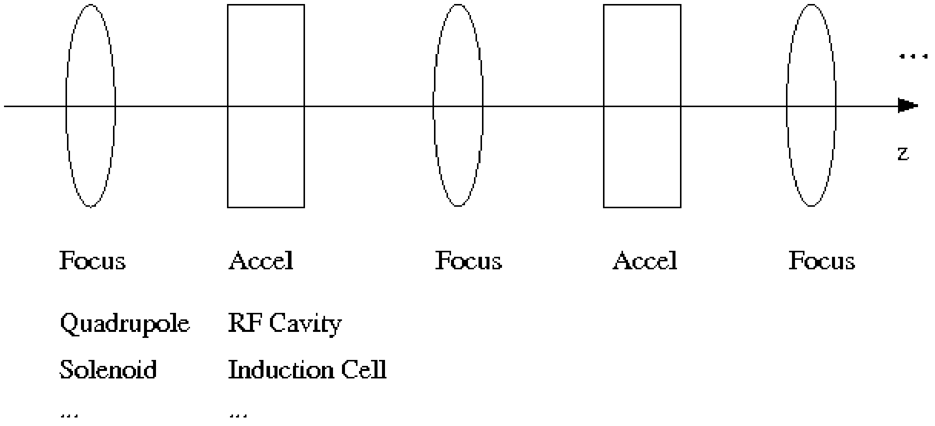


### Induction Cell



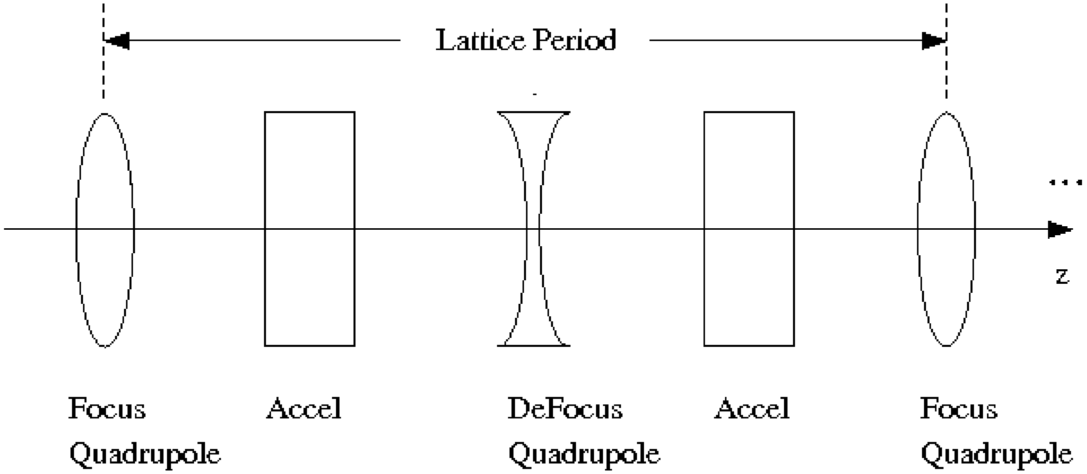
# Machine Lattice

Applied field structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:

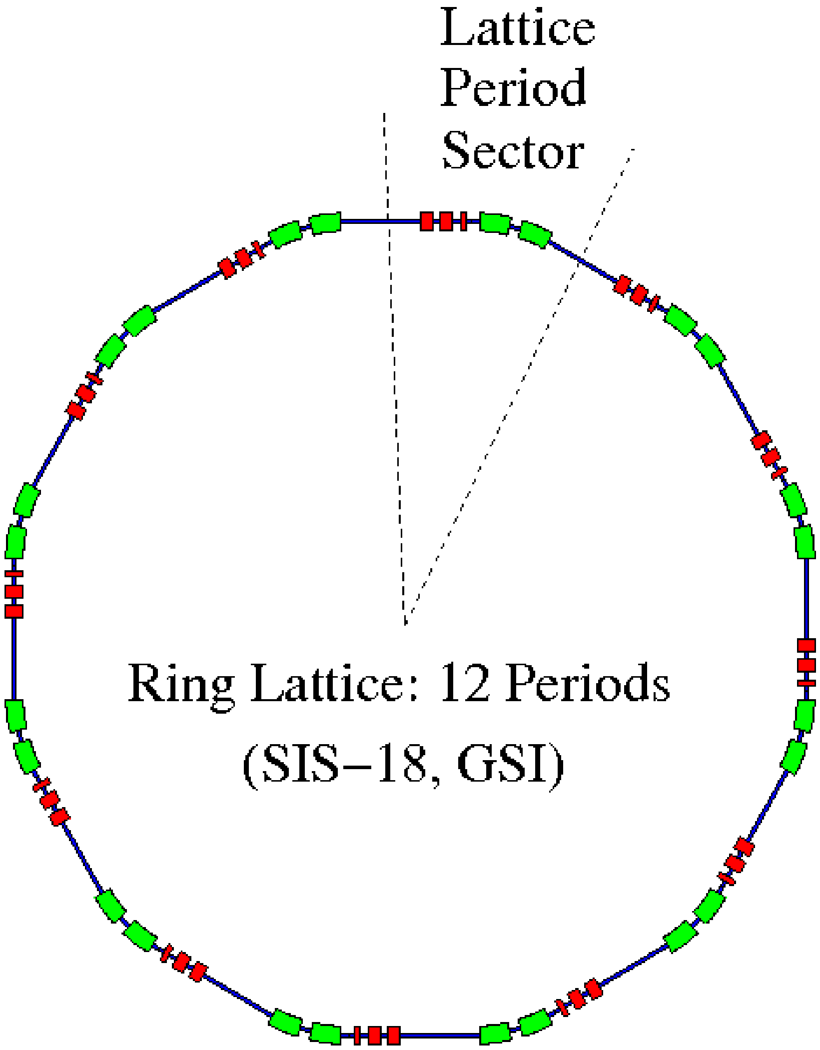


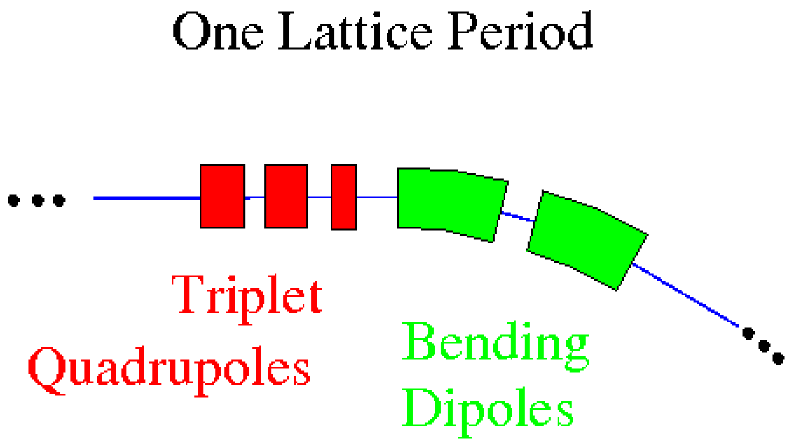
- Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



Lattices for rings and some beam insertion/extraction sections also incorporate bends and more complicated periodic structures:





- Elements to insert beam into and out of ring further complicate lattice
- Acceleration cells also present
  (typically several RF cavities at one or more location)

# S3: Description of Applied Focusing Fields

## S3A: Overview

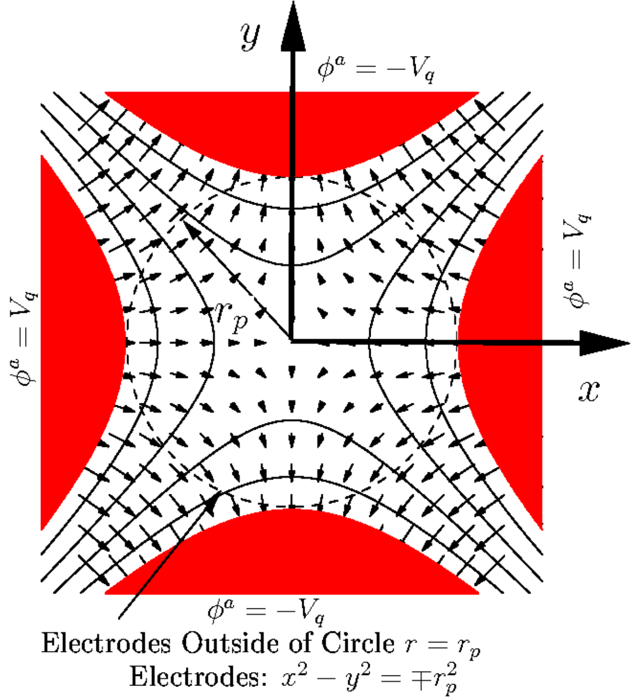
Applied fields for focusing, bending, and acceleration enter the equations of motion via:

$$\mathbf{E}^a = \text{Applied Electric Field}$$

$$\mathbf{B}^a = \text{Applied Magnetic Field}$$

Generally, these fields are produced by sources (often static or slowly varying in time) located outside an aperture or so-called pipe radius $r = r_p$ . For example, the electric and magnetic quadrupoles of S2:

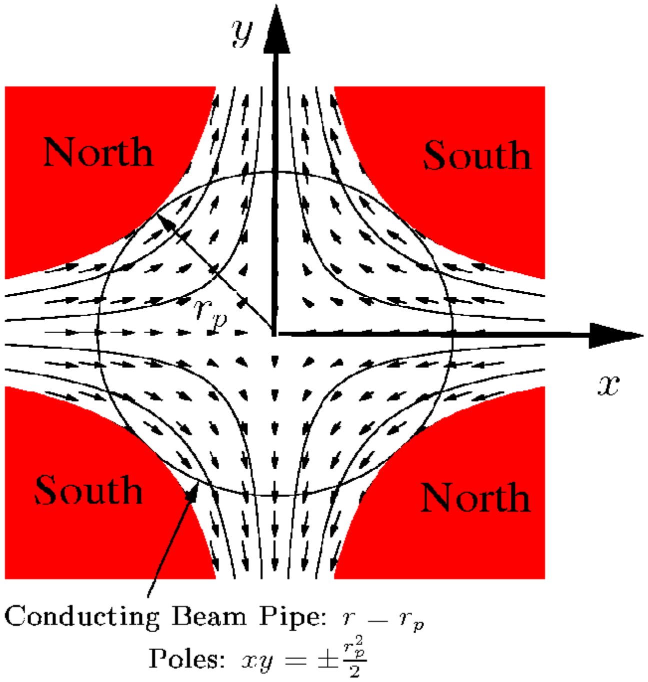
Electric Quadrupole



Hyperbolic material surfaces outside pipe radius

$$r = r_p$$

Magnetic Quadrupole



The fields of such classes of magnets obey the vacuum Maxwell Equations within the aperture:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E}^a &= 0 & \qquad \nabla \cdot \mathbf{B}^a &= 0 \\ \nabla \times \mathbf{E}^a &= -\frac{\partial}{\partial t}\mathbf{B}^a & \qquad \nabla \times \mathbf{B}^a &= \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^a \end{aligned}}$$

If the fields are static or sufficiently slowly varying (quasistatic) where the time derivative terms can be neglected, then the fields in the aperture will obey the static vacuum Maxwell equations:

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{E}^a &= 0 & \qquad \nabla \cdot \mathbf{B}^a &= 0 \\ \nabla \times \mathbf{E}^a &= 0 & \qquad \nabla \times \mathbf{B}^a &= 0 \end{aligned}}$$

In general, optical elements are tuned to limit the strength of nonlinear field terms so the beam experiences primarily linear applied fields.

- Linear fields allow better preservation of beam quality

Removal of *all* nonlinear fields cannot be accomplished

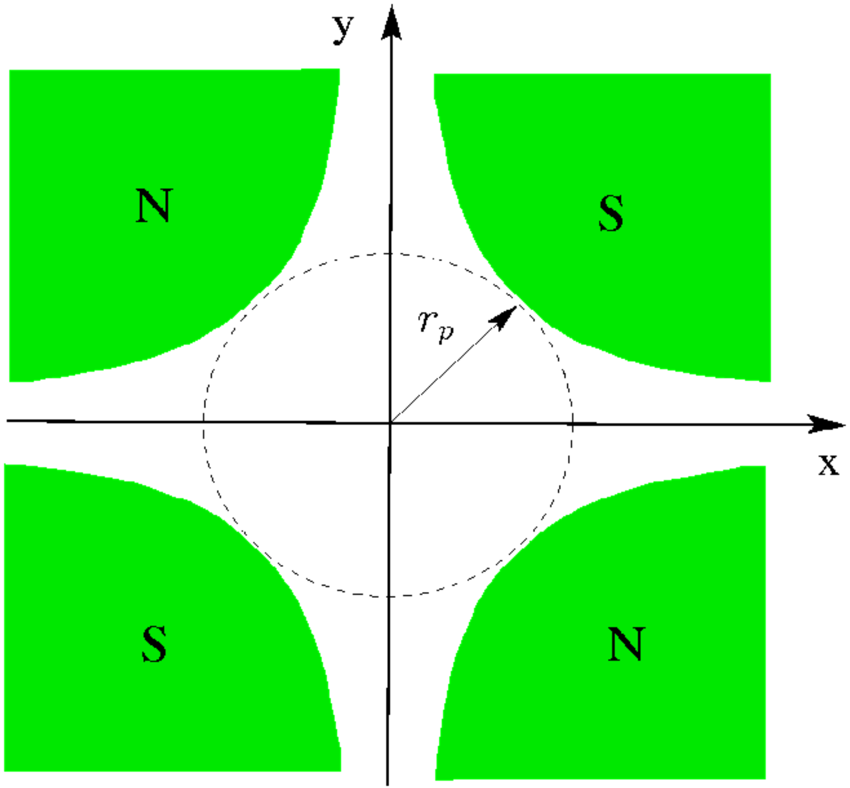
- 3D structure of the Maxwell equations precludes for finite geometry optics
- Even in finite geometries deviations from optimal structures and symmetry will result in nonlinear fields

As an example of this, when an ideal 2D iron magnet with infinite hyperbolic poles is truncated radially for finite 2D geometry, this leads to nonlinear focusing fields even in 2D:

- Truncation necessary along with confinement of return flux in yoke

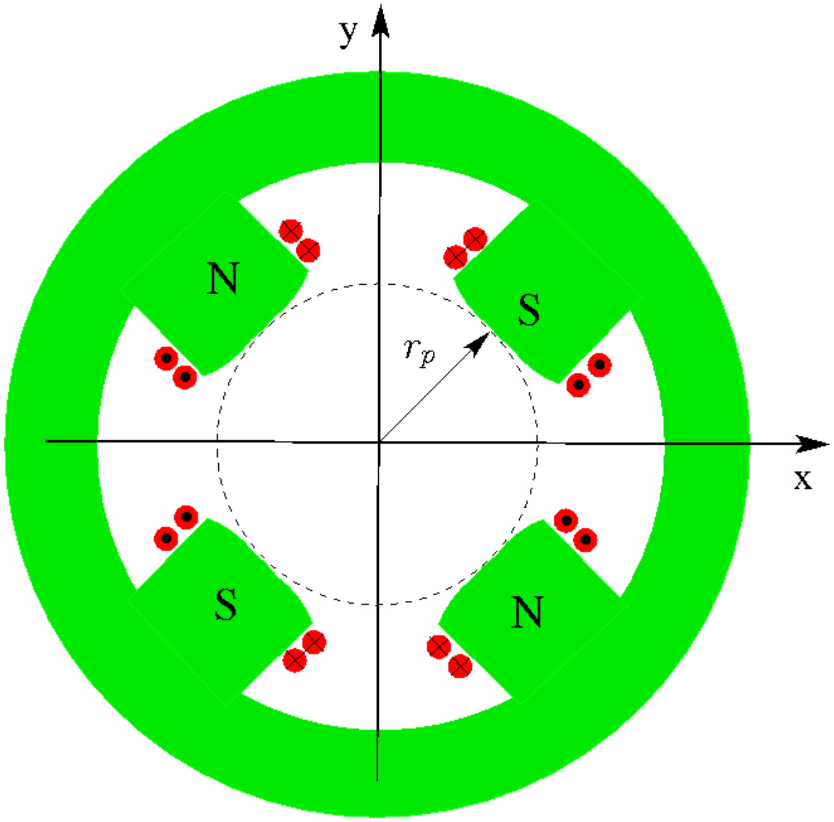
## Cross-Sections of Iron Quadrupole Magnets

### Ideal (infinite geometry)



Hyperbolic Iron Pole Sections
(infinite)

### Practical (finite geometry)



Shaped Iron Pole Sections
(finite)

The design of optimized electric and magnetic optics for accelerators is a specialized topic with a vast literature. It is not be possible to cover this topic in this brief survey. In this section we will overview a limited subset of material on magnetic optics including:

- (see: S3B) Magnetic field expansions for focusing and bending
- (see: S3C) Hard edge equivalent models
- (see: S3D) 2D multipole models and nonlinear field scalings
- (see: S3E) Good field radius

Much of the material presented can be immediately applied to static Electric Optics since the vacuum Maxwell equations are the same for static Electric $\mathbf{E}^a$ and Magnetic $\mathbf{B}^a$ fields in vacuum.

# S3B: Magnetic Field Expansions for Focusing and Bending

Forces from transverse $(B_z^a = 0)$ magnetic fields enter the transverse equations of motion via:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}^a$$

Force: $\mathbf{F}_\perp^a \simeq q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}_\perp^a$

$$\mathbf{v} \simeq \beta_b c\hat{\mathbf{z}}$$

Field: $\mathbf{B}_\perp^a = \hat{\mathbf{x}}B_x^a + \hat{\mathbf{y}}B_y^a$

Combined these give:

$$\boxed{\begin{aligned} F_x^a &\simeq -q\beta_b c B_y^a \\ F_y^a &\simeq \quad q\beta_b c B_x^a \end{aligned}}$$

Field components entering these expressions can be expanded about $\mathbf{x}_\perp = 0$

- Element center and design orbit taken to be at $\mathbf{x}_\perp = 0$

$$B_x^a = \overset{1}{B_x^a}(0) + \overset{2}{\frac{\partial B_x^a}{\partial y}}(0)y + \overset{3}{\frac{\partial B_x^a}{\partial x}}(0)x$$

Nonlinear Focus

$$+ \frac{1}{2}\frac{\partial^2 B_x^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_x^a}{\partial x \partial y}(0)xy + \frac{1}{2}\frac{\partial B_x^a}{\partial y^2}(0)y^2 + \cdots$$

$$B_y^a = \overset{1}{B_y^a}(0) + \overset{2}{\frac{\partial B_y^a}{\partial x}}(0)x + \overset{3}{\frac{\partial B_y^a}{\partial y}}(0)y$$

Nonlinear Focus

$$+ \frac{1}{2}\frac{\partial^2 B_y^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_y^a}{\partial x \partial y}(0)xy + \frac{1}{2}\frac{\partial B_y^a}{\partial y^2}(0)y^2 + \cdots$$

Terms:
1: Dipole Bend
2: Normal Quad Focus
3: Skew Quad Focus

Sources of undesired nonlinear applied field components include:

- Intrinsic finite 3D geometry and the structure of the Maxwell equations
- Systematic errors or sub-optimal geometry associated with practical trade-offs in fabricating the optic
- Random construction errors in individual optical elements
- Alignment errors of magnets in the lattice giving field projections in unwanted directions
- Excitation errors effecting the field strength
  - Currents in coils not correct and/or unbalanced

More advanced treatments exploit less simple power-series expansions to express symmetries more clearly:

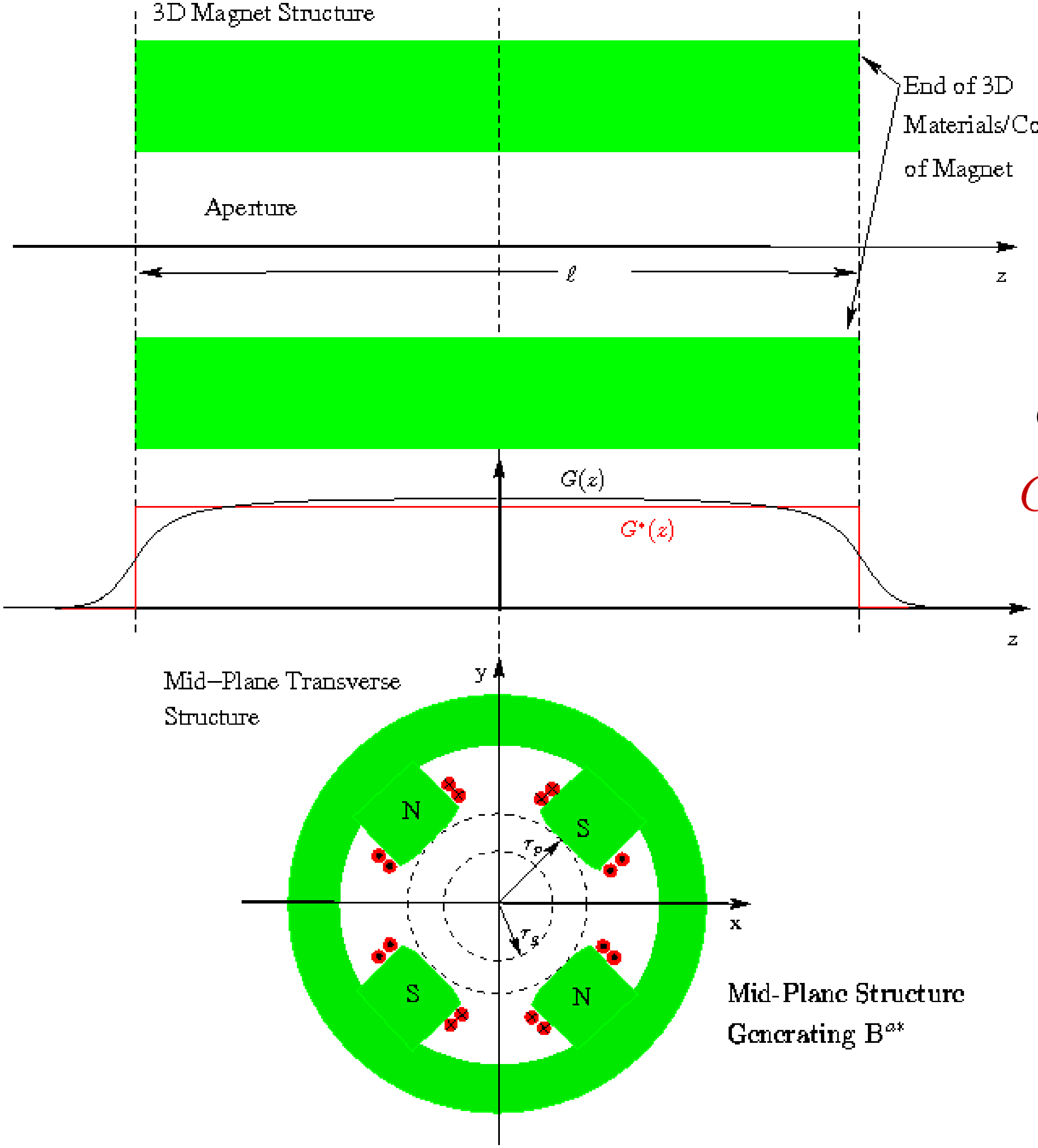
- Maxwell equations constrain structure of solutions
  - Expansion coefficients are NOT all independent
- Forms appropriate for bent coordinate systems in dipole bends can become complicated

# S3C: Hard Edge Equivalent Models

Real 3D magnets can often be modeled with sufficient accuracy by 2D hard-edge "equivalent" magnets that give the same approximate focusing impulse to the particle as the full 3D magnet

- Objective is to provide same approximate applied focusing "kick" to particles with different focusing gradient functions $G(s)$

See Figure Next Slide



$G(z)$ = 3D Field Gradient

$G^*(z)$ = Hard-Edge Equivalent Field Gradient

Many prescriptions exist for calculating the effective axial length and strength of hard-edge equivalent models

- See Review: Lund and Bukh, PRSTAB 7 204801 (2004), Appendix C

Here we overview a simple equivalence method that has been shown to work well:

For a relatively long, but finite axial length magnet with 3D gradient function:

$$G(z) \equiv \left. \frac{\partial B_x^a}{\partial y} \right|_{x=y=0}$$

Take hard-edge equivalent parameters:

- Take $z = 0$ at the axial magnet mid-plane

Gradient: $$G^* \equiv G(z=0)$$

Axial Length: $$\ell \equiv \frac{1}{G(z=0)} \int_{-\infty}^{\infty} dz\, G(z)$$

- More advanced equivalences can be made based more on particle optics
  - Disadvantage of such methods is “equivalence” changes with particle energy and must be revisited as optics are tuned

# S3D: 2D Transverse Multipole Magnetic Fields

In many cases, it is sufficient to characterize the field errors in 2D hard-edge equivalent as:

$$\overline{B_x}(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\ B_x^a(x,y,z)$$

$$\overline{B_y}(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\ B_y^a(x,y,z)$$

2D Effective Fields

3D Fields

Operating on the vacuum Maxwell equations with: $\displaystyle\int_{-\infty}^{\infty}\frac{dz}{\ell}\cdots$

yields the (exact) 2D Transverse Maxwell equations :

$$\frac{\partial\overline{B_x}(x,y)}{\partial y} = \frac{\partial\overline{B_y}(x,y)}{\partial x} \qquad \Leftarrow \text{ From } \quad \nabla\times\mathbf{B}^a = 0$$

$$\frac{\partial\overline{B_x}(x,y)}{\partial x} = -\frac{\partial\overline{B_y}(x,y)}{\partial y} \qquad \Leftarrow \text{ From } \quad \nabla\cdot\mathbf{B}^a = 0$$

These equations are recognized as the Cauchy-Riemann conditions for a complex field variable:

$$\boxed{\underline{B}^* \equiv \overline{B}_x - i\overline{B}_y \qquad i \equiv \sqrt{-1}}$$

to be an analytical function of the complex variable:

$$\boxed{\underline{z} \equiv x + iy \qquad i \equiv \sqrt{-1}}$$

Notation:
Underlines denote complex variables where confusion may arise

Cauchy-Riemann Conditions

2D Magnetic Field

$$\underline{F} = u(x,y) + iv(x,y) \qquad\qquad u = \overline{B}_x \quad v = -\overline{B}_y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \longrightarrow \quad \frac{\partial \overline{B}_x(x,y)}{\partial x} = -\frac{\partial \overline{B}_y(x,y)}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \longrightarrow \quad \frac{\partial \overline{B}_x(x,y)}{\partial y} = \frac{\partial \overline{B}_y(x,y)}{\partial x}$$

$\underline{F} = u + iv$ analytic func of $\underline{z} = x + iy$

$\underline{F} = \overline{B}_x - i\overline{B}_y$ analytic func of $\underline{z} = x + iy$

Note the complex field which is an analytic function of $\underline{z} = x + iy$ is $\underline{B}^* = \overline{B}_x - i\overline{B}_y$ NOT $\underline{B} = \overline{B}_x + i\overline{B}_y$ . This is *not* a typo and is necessary for $\underline{B}^*$ to satisfy the Cauchy-Riemann conditions.

- See problem sets for illustration

It follows that $\underline{B}^*(\underline{z})$ can be analyzed using the full power of the highly developed theory of analytical functions of a complex variable.

Expand $\underline{B}^*(\underline{z})$ as a Laurent Series within the vacuum aperture as:

$$\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1}$$

$$\underline{b}_n = \text{const} \;\; \text{(complex)}$$

$$n = \text{Multipole Index}$$

The $\underline{b}_n$ are called “multipole coefficients” and give the structure of the field. The multipole coefficients can be resolved into real and imaginary parts as:

$$\boxed{\begin{aligned} &\underline{b}_n = \mathcal{A}_n - i\mathcal{B}_n \\ &\qquad\qquad \mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles} \\ &\qquad\qquad \mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles} \end{aligned}}$$

Some algebra identifies the polynomial symmetries of low-order terms as:

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}$

| Index | Name | Normal ($\mathcal{A}_n = 0$) | | Skew ($\mathcal{B}_n = 0$) | |
|---|---|---|---|---|---|
| $n$ | | $\overline{B_x}/\mathcal{B}_n$ | $\overline{B_y}/\mathcal{B}_n$ | $\overline{B_x}/\mathcal{A}_n$ | $\overline{B_y}/\mathcal{A}_n$ |
| 1 | Dipole | $0$ | $1$ | $1$ | |
| 2 | Quadrupole | $y$ | $x$ | $x$ | $-y$ |
| 3 | Sextupole | $2xy$ | $x^2 - y^2$ | $x^2 - y^2$ | $-2xy$ |
| 4 | Octupole | $3x^2y - y^3$ | $x^3 - 3xy^2$ | $x^3 - 3xy^2$ | $-3x^2y + y^3$ |
| 5 | Decapole | $4x^3y - 4xy^3$ | $x^4 - 6x^2y^2 + y^4$ | $x^4 - 6x^2y^2 + y^4$ | $-4x^3y + 4xy^3$ |

Comments:

- Reason for pole names most apparent from polar representation (see following pages) and sketches of the magnetic pole structure
- Caution: In so-called “US notation”, poles are labeled with index *n* -> *n* -1
    - Arbitrary in 2D but US choice *not* good notation in 3D generalizations

## Comments continued:

- Normal and Skew symmetries can be taken as a symmetry *definition.* But this choice makes sense for $n = 2$ quadrupole focusing terms:

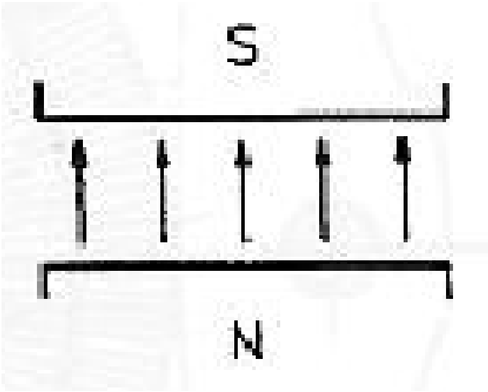
$$\overline{F_x^a} = -q\beta_b c\overline{B_y} \quad = -q\beta_b c(\mathcal{B}_2 x - \mathcal{A}_2 y)$$
$$\overline{F_y^a} = \quad q\beta_b c\overline{B_x} \quad = \quad q\beta_b c(\mathcal{B}_2 y + \mathcal{A}_2 x)$$

In equations of motion:

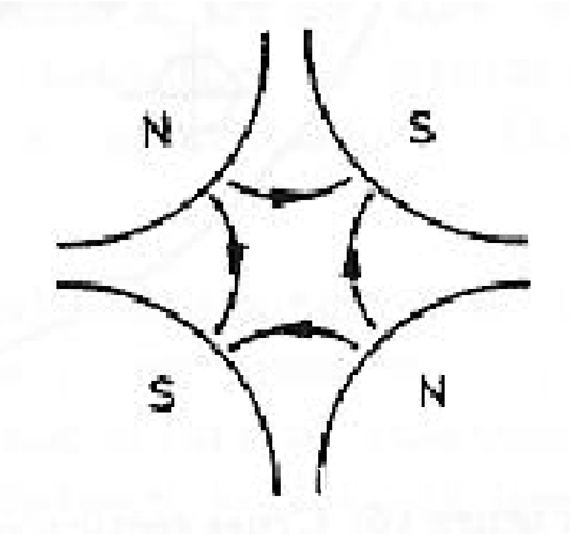
| | | | | |
|---|---|---|---|---|
| Normal | $\Rightarrow$ | $\mathcal{B}_2$: | $x$-eqn, $x$-focus | $y$-eqn, $y$-defocus |
| Skew | $\Rightarrow$ | $\mathcal{A}_2$: | $x$-eqn, $y$-defocus | $y$-eqn, $x$-defocus |

## Magnetic Pole Symmetries (normal orientation):

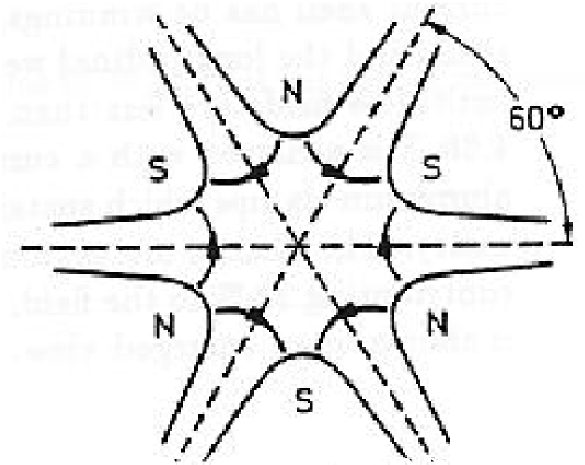
Dipole (n=1)



Quadrupole (n=2)



Sextupole (n=3)



- Actively rotate normal field structures clockwise through an angle of $\pi/(2n)$ for skew field component symmetries

## Multipole scale/units

Frequently, in the multipole expansion:

$$\boxed{\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1}}$$

the multipole coefficients $\underline{b}_n$ are rescaled as

$$\underline{b}_n \rightarrow \underline{b}_n r_p^{n-1}$$

$r_p = \text{Aperture ”Pipe” Radius}$

Closest radius of approach of magnetic sources and/or aperture materials

so that the expansions becomes

$$\boxed{\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}}$$

Advantages of alternative notaiton:

- Multipoles $\underline{b}_n$ given directly in field units regardless of index *n*
- Scaling of field amplitudes with radius within the magnet bore becomes clear

## Scaling of Fields produced by multipole term:

Higher order multipole coefficients (larger *n* values) leading to nonlinear focusing forces decrease rapidly within the aperture. To see this use a polar representation for $\underline{z}$, $\underline{b}_n$

$$\boxed{\begin{array}{ll} \underline{z} = x + iy = re^{i\theta} & r = \sqrt{x^2 + y^2} \\ & \theta = \arctan[y, x] \\ \underline{b}_n = |\underline{b}_n| e^{i\psi_n} & \psi_n = \text{Real Const} \end{array}}$$

Thus, the nth order multipole terms scale as

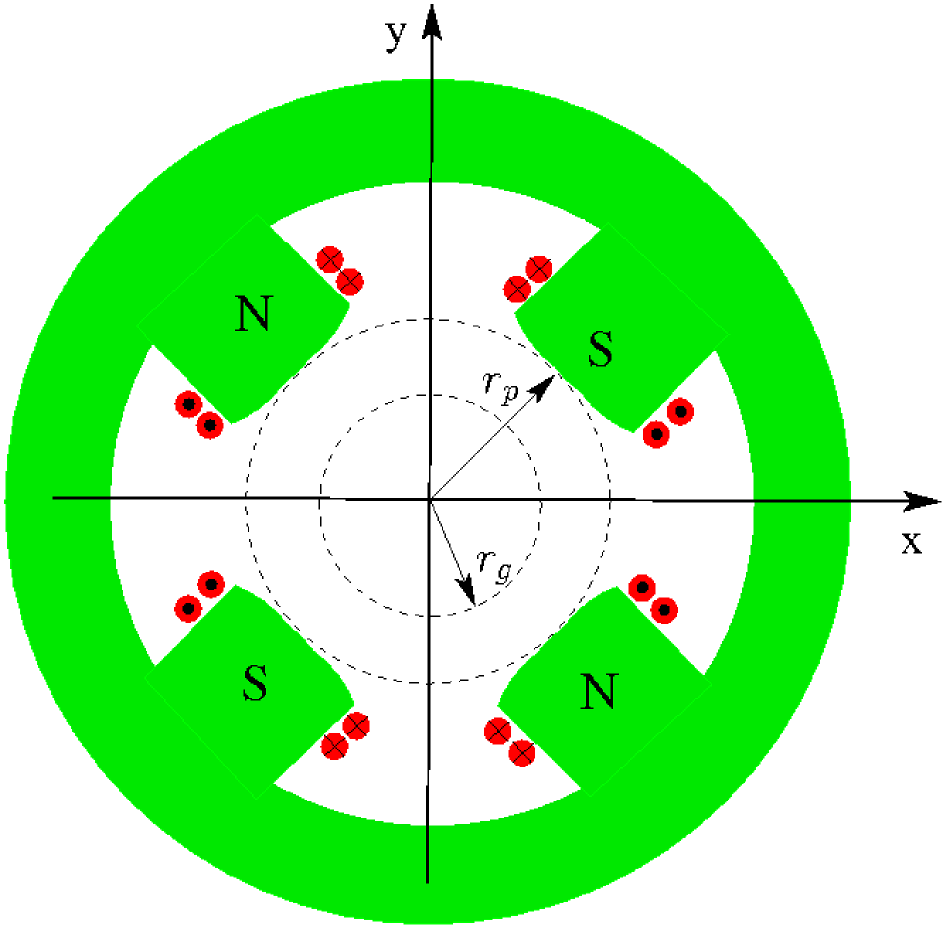
$$\boxed{\underline{b}_n \left( \frac{\underline{z}}{r_p} \right)^{n-1} = |\underline{b}_n| \left( \frac{r}{r_p} \right)^{n-1} e^{i[(n-1)\theta + \psi_n]}}$$

- Unless the coefficient $|\underline{b}_n|$ is very large, high order terms in *n* will become small rapidly as $r_p$ decreases
- Better field quality can be obtained for a given magnet design by simply making the clear bore $r_p$ larger, or alternatively using smaller bundles (more tight focus) of particles
  - Larger bore machines/magnets cost more. So designs become trade-off between cost and performance.
  - Stronger focusing to keep beam from aperture can be unstable

# S3E: Good Field Radius

Often a magnet design will have a so-called "good-field" radius $r = r_g$ that the maximum field errors are specified on.

- In superior designs the good field radius can be around ~70% or more of the clear bore aperture to the beginning of material structures of the magnet.
- Beam particles should evolve with radial excursions with $r < r_g$



$$r_p = \text{Clear Bore Radius}$$
$$\sim \text{Pole Radius Typical}$$

$$r_g = \text{Good Field Radius}$$
$$\sim 70\%\ r_p\ \text{Typical}$$

Comments:

- Particle orbits are designed to remain within radius $r_g$
- Field error statements are readily generalized to 3D since:

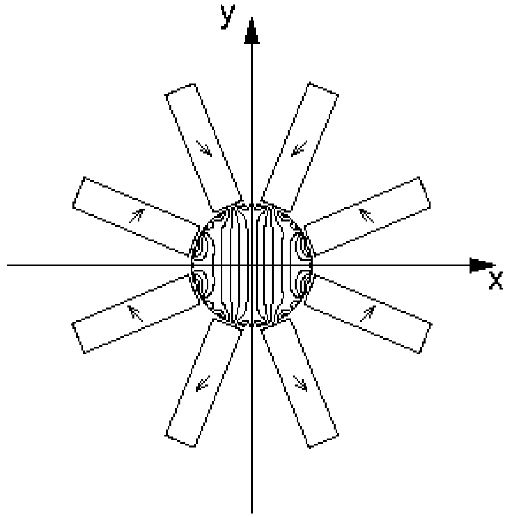
$$\begin{aligned} \nabla \cdot \mathbf{B}^a &= 0 \\ \nabla \times \mathbf{B}^a &= 0 \end{aligned} \quad \Longrightarrow \quad \nabla^2 \mathbf{B}^a = 0$$

and therefore each component of $\mathbf{B}^a$ satisfies a Laplace equation within the vacuum aperture. Therefore, field errors decrease when moving more deeply within a source-free region.

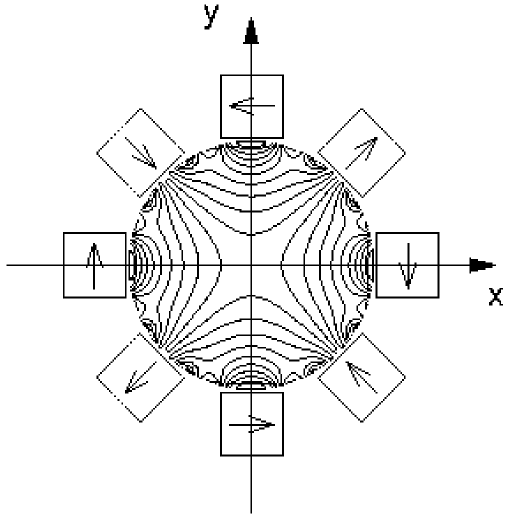
# S3F: Example Permanent Magnet Assemblies

A few examples of practical permanent magnet assemblies with field contours are provided to illustrate error field structures in practical devices

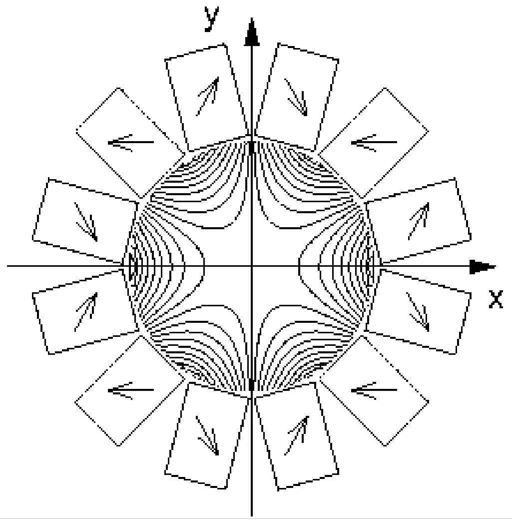
8 Rectangular Block Dipole



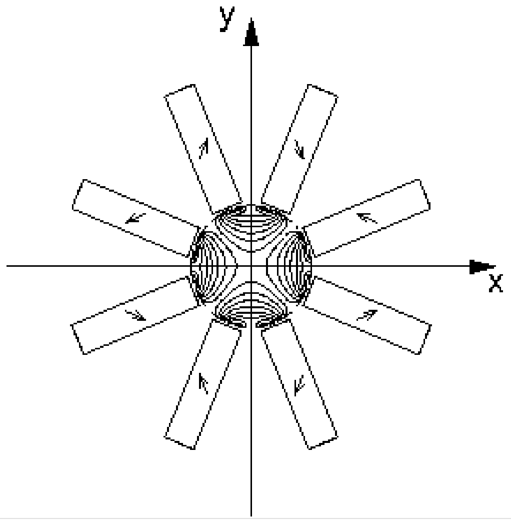
8 Square Block Quadrupole



12 Rectangular Block Sextupole



8 Rectangular Block Quadrupole



For more info on permanent magnet design see: Lund and Halbach, Fusion Engineering Design, **32-33**, 401-415 (1996)

# Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/ap_2018/

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